A NOTE ON DIRAC SPINORS IN A NON-FLAT SPACE-TIME OF GENERAL RELATIVITY.

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Abstract. Some aspects of Dirac spinors are resumed and studied in order to interpret mathematically the P and T operations in a gravitational field.

1. Algebra and geometry of two-component spinors.

From a mathematical point of view two-component spinors naturally arise when one tries to understand geometrically the well-known group homomorphism

$$\varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3,\mathbb{R}) \tag{1.1}$$

given by the following explicit formulas:

$$\begin{aligned}
&S^0_0=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_1+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_2}{2},\\
&S^0_1=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_2+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_1}{2},\\
&S^0_2=\frac{\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_1-\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_2}{2\,i},\\
&S^0_3=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_1-\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_2}{2},
\end{aligned} \tag{1.2}$$

$$\begin{aligned}
&S^1_0=\frac{\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_1+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_2}{2},\\
&S^1_1=\frac{\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_1+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_2}{2},\\
&S^1_2=\frac{\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_1-\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_2}{2\,i},\\
&S^1_3=\frac{\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_2-\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_2}{2},
\end{aligned} \tag{1.3}$$

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$$
\begin{aligned}
S^2_0&=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_2-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_2}{2\,i},\\
S^2_1&=\frac{\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_2-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_1}{2\,i},\\
S^2_2&=\frac{\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_2-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_2-\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_1}{2},\\
S^2_3&=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^1_2-\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^2_2}{2\,i},
\end{aligned}
\tag{1.4}
$$

$$
\begin{aligned}
S^3_0&=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_2-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_2}{2},\\
S^3_1&=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_1-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_2-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_1}{2},\\
S^3_2&=\frac{\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_1-\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_2+\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_2-\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_1}{2\,i},\\
S^3_3&=\frac{\overline{\mathfrak{S}^1_1}\,\mathfrak{S}^1_1+\overline{\mathfrak{S}^2_2}\,\mathfrak{S}^2_2-\overline{\mathfrak{S}^2_1}\,\mathfrak{S}^2_1-\overline{\mathfrak{S}^1_2}\,\mathfrak{S}^1_2}{2}
\end{aligned}
\tag{1.5}
$$

(see [1], [2], and [3], see also [4] and [5] for more details). Here in (1.2), (1.3), (1.4), and (1.5) by $\mathfrak{S}^i_j$ we denote the components of a 2×2 complex matrix $\mathfrak{S}\in\operatorname{SL}(2,\mathbb{C})$, while S^i_j are the components of the matrix $S=\varphi(\mathfrak{S})$ produced from $\mathfrak{S}$ by applying the homomorphism (1.1).

Let M be a *space-time* manifold, i. e. this is a 4-dimensional orientable manifold equipped with a pseudo-Euclidean metric $\mathbf{g}$ of the Minkowski-type signature $(+,-,-,-)$ and carrying a special smooth geometric structure which is called a *polarization*. Once some polarization is fixed, one can distinguish the *Future light cone* from the *Past light cone* at each point $p\in M$ (see [6] for more details). A moving frame $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ of the tangent bundle TM is an ordered set of four smooth vector fields $\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3$ which are defined and $\mathbb{R}$-linearly independent at each point p of the open set $U\subset M$. This moving frame is called a *positively polarized right orthonormal frame* if the following conditions are fulfilled:

(1) the value of the first vector filed $\boldsymbol{\Upsilon}_0$ at each point $p\in U$ belongs to the interior of the Future light cone determined by the polarization of M;
(2) it is a right frame in the sense of the orientation of M;
(3) the metric tensor $\mathbf{g}$ is given by the standard Minkowski matrix in this frame:

$$
g_{ij}=g(\boldsymbol{\Upsilon}_i,\boldsymbol{\Upsilon}_j)=\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix}.
\tag{1.6}
$$

Assume that we have some other positively polarized right orthonormal frame $(\tilde U,\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ such that $U\cap\tilde U\neq\varnothing$. Then at each point $p\in U\cap\tilde U$ we can write the following relationships for the frame vectors:

$$
\tilde{\boldsymbol{\Upsilon}}_i=\sum_{j=0}^{3}S^j_i\,\boldsymbol{\Upsilon}_j,\qquad\qquad \boldsymbol{\Upsilon}_i=\sum_{j=0}^{3}T^j_i\,\tilde{\boldsymbol{\Upsilon}}_j.
\tag{1.7}
$$

The relationships (1.7) are called *transition formulas*, while the coefficients S_i^j and T_i^j in them are the components of two mutually inverse transition matrices S and T. Since both frames $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ are positively polarized right orthonormal frames, the transition matrices S and T both are orthochronous Lorentzian matrices with $\det S = 1$ and $\det T = 1$. Such matrices form the *special orthochronous matrix Lorentz group* $\mathrm{SO}^+(1,3,\mathbb{R})$ (see [6] for more details).

Let SM be a two-dimensional smooth complex vector bundle over the space-time M equipped with a non-vanishing skew-symmetric spin-tensorial field $\mathbf{d}$. This spin-tensorial field $\mathbf{d}$ is called the *spin-metric tensor*. A moving frame $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ of SM is an ordered set of two smooth section $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ of SM over the open set U which are $\mathbb{C}$-linearly independent at each point $p \in U$. A moving frame $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ is called an *orthonormal frame* if

$$d_{ij} = d(\boldsymbol{\Psi}_i, \boldsymbol{\Psi}_j) = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \tag{1.8}$$

i. e. if the spin-metric tensor $\mathbf{d}$ is given by the skew-symmetric matrix (1.8) in this frame. Assume that we have two orthonormal frames $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ of the bundle SM with overlapping domains $U \cap \tilde{U} \neq \varnothing$. Then at each point $p \in U \cap \tilde{U}$ we can write the following transition formulas:

$$\tilde{\boldsymbol{\Psi}}_i = \sum_{j=1}^{2} \mathfrak{S}_i^j\, \boldsymbol{\Psi}_j, \qquad\qquad \boldsymbol{\Psi}_i = \sum_{j=1}^{2} \mathfrak{T}_i^j\, \tilde{\boldsymbol{\Psi}}_j. \tag{1.9}$$

Since both frames $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ are orthonormal with respect to spin-metric tensor $\mathbf{d}$, both transition matrices $\mathfrak{S}$ and $\mathfrak{T} = \mathfrak{S}^{-1}$ with the components $\mathfrak{S}_i^j$ and $\mathfrak{T}_i^j$ in (1.9) belong to the special linear group $\mathrm{SL}(2,\mathbb{C})$.

Definition 1.1. A two-dimensional complex vector bundle SM over the space-time manifold M equipped with a nonzero spin-metric $\mathbf{d}$ is called a *spinor bundle* if each orthonormal frame $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ of SM is associated with some positively polarized right orthonormal frame $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ of the tangent bundle TM such that for any two orthonormal frames $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ with overlapping domains $U \cap \tilde{U} \neq \varnothing$ the associated tangent frames $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ are related to each other by means of the formulas (1.7), where the transition matrices S and T are obtained from the transition matrices $\mathfrak{S}$ and $\mathfrak{T}$ in (1.9) by applying the homomorphism (1.1), i. e. $S = \varphi(\mathfrak{S})$ and $T = \varphi(\mathfrak{T})$.

2. An algebraic background for Dirac spinors.

The group homomorphism (1.1) is an algebraic background for two-component spinors. They form a complex bundle over M introduced by the definition 1.1. In order to construct an algebraic background for Dirac spinors we need to extend the group homomorphism (1.1) to bigger groups. For the group $\mathrm{SO}^+(1,3,\mathbb{R})$ in (1.1) we have the following natural enclosure:

$$\mathrm{SO}^+(1,3,\mathbb{R}) \subset \mathrm{O}(1,3,\mathbb{R}). \tag{2.1}$$

The complete matrix Lorentzian group $\mathrm{O}(1,3,\mathbb{R})$ in (2.1) is generated by adding the following two matrices to $\mathrm{SO}^{+}(1,3,\mathbb{R})$:

$$P=\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix},\qquad T=\begin{Vmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{Vmatrix}.\tag{2.2}$$

The first matrix P in (2.2) is the *spatial inversion matrix*. The second matrix T is the *time inversion matrix*. The matrices P and T in (2.2) are commutating:

$$P\cdot T=T\cdot P=-\mathbf{1}.\tag{2.3}$$

Apart from (2.3), we have the following relationships:

$$P^2=\mathbf{1},\qquad T^2=\mathbf{1}\tag{2.4}$$

Due to (2.3) and (2.4) each matrix $\hat{S}$ of the group $\mathrm{O}(1,3,\mathbb{R})$ is represented as

$$\hat{S}=S\quad\text{or}\quad\hat{S}=P\cdot S\quad\text{or}\quad\hat{S}=T\cdot S\quad\text{or}\quad\hat{S}=-S,\tag{2.5}$$

where $S\in\mathrm{SO}^{+}(1,3,\mathbb{R})$. Using the representation (2.5), we can reduce the multiplication in $\mathrm{O}(1,3,\mathbb{R})$ to the multiplication in $\mathrm{SO}^{+}(1,3,\mathbb{R})$:

	S_2	$P\cdot S_2$	$T\cdot S_2$	$-S_2$
S_1	$S_1\cdot S_2$	$P\cdot(S_3\cdot S_2)$	$T\cdot(S_3\cdot S_2)$	$-(S_1\cdot S_2)$
$P\cdot S_1$	$P\cdot(S_1\cdot S_2)$	$S_3\cdot S_2$	$-(S_3\cdot S_2)$	$T\cdot(S_1\cdot S_2)$
$T\cdot S_1$	$T\cdot(S_1\cdot S_2)$	$-(S_3\cdot S_2)$	$S_3\cdot S_2$	$P\cdot(S_1\cdot S_2)$
$-S_1$	$-(S_1\cdot S_2)$	$T\cdot(S_3\cdot S_2)$	$P\cdot(S_3\cdot S_2)$	$S_1\cdot S_2$

The matrix $S_3\in\mathrm{SO}^{+}(1,3,\mathbb{R})$ in the above table is determined by the formula

$$S_3=P\cdot S_1\cdot P.\tag{2.6}$$

Since $P^2=1$, we have $P=P^{-1}$ and we can write (2.6) as $S_3=\psi(S_1)$, where $\psi\colon\ \mathrm{SO}^{+}(1,3,\mathbb{R})\to\mathrm{SO}^{+}(1,3,\mathbb{R})$ is the group homomorphism given by the formula

$$S\mapsto S'=\psi(S)=P\cdot S\cdot P^{-1}.\tag{2.7}$$

Assume that the matrix S is obtained by means of the group homomorphism (1.1), i. e. assume that $S=\varphi(\mathfrak{S})$, where $\mathfrak{S}\in\mathrm{SL}(2,\mathbb{C})$. Then $S\in\mathrm{SO}^{+}(1,3,\mathbb{R})$, and $\psi(S)\in\mathrm{SO}^{+}(1,3,\mathbb{R})$, therefore $\psi(S)=\varphi(\mathfrak{S}')$ for some matrix $\mathfrak{S}\in\mathrm{SL}(2,\mathbb{C})$ since φ is a surjective homomorphism. The matrix $\mathfrak{S}'$ is determined uniquely up to the sign. By means of direct calculations we derive

$$\mathfrak{S}'=\pm\,(\mathfrak{S}^{-1})^{\dagger},\tag{2.8}$$

i. e. $\mathfrak{S}'$ is the Hermitian conjugate matrix for $\mathfrak{S}^{-1}$. Choosing the plus sign in (2.8), we obtain one more group homomorphism

$$\mathfrak{S} \mapsto \mathfrak{S}' = \psi'(\mathfrak{S}) = (\mathfrak{S}^{-1})^{\dagger}. \tag{2.9}$$

The homomorphisms (1.1), (2.7), and (2.9) compose the commutative diagram

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\ \varphi\ } & S \\ {\scriptstyle\psi'}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ \mathfrak{S}' & \xrightarrow{\ \varphi\ } & S' \end{array}$$

Now let's remember the construction of the homomorphism (1.1), see [1], [2], [3], or [4]. It is constructed on the base of the equality

$$\mathfrak{S} \cdot \boldsymbol{\sigma}_m \cdot \mathfrak{S}^{\dagger} = \sum_{k=0}^{3} S_m^k \, \boldsymbol{\sigma}_k, \tag{2.10}$$

where $\boldsymbol{\sigma}_0$ is the unit matrix, while $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$ are the well-known Pauli matrices:

$$\begin{aligned} &\boldsymbol{\sigma}_0 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, && \boldsymbol{\sigma}_2 = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}, \\ &\boldsymbol{\sigma}_1 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, && \boldsymbol{\sigma}_3 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}. \end{aligned} \tag{2.11}$$

Lemma 2.1. *For $\mathfrak{S} \in \mathrm{SL}(2, \mathbb{C})$ the relationship (2.10) can be transformed to*

$$(\mathfrak{S}^{-1})^{\dagger} \cdot \tilde{\boldsymbol{\sigma}}_m \cdot \mathfrak{S}^{-1} = \sum_{k=0}^{3} S_m^k \, \tilde{\boldsymbol{\sigma}}_k, \tag{2.12}$$

where $\tilde{\boldsymbol{\sigma}}_m = \varepsilon_m \, \boldsymbol{\sigma}_m^{-1}$ and $\varepsilon_m = \det(\boldsymbol{\sigma}_m)$, i. e. they are given by the formulas

$$\begin{aligned} &\tilde{\boldsymbol{\sigma}}_0 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, && \tilde{\boldsymbol{\sigma}}_2 = \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}, \\ &\tilde{\boldsymbol{\sigma}}_1 = \begin{Vmatrix} 0 & -1 \\ -1 & 0 \end{Vmatrix}, && \tilde{\boldsymbol{\sigma}}_3 = \begin{Vmatrix} -1 & 0 \\ 0 & 1 \end{Vmatrix}. \end{aligned} \tag{2.13}$$

Proof. Note that for a 2×2 matrix A with the unit determinant $\det A = 1$ we have

$$A^{-1} = \begin{Vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{Vmatrix}^{-1} = \begin{Vmatrix} a_2^2 & -a_2^1 \\ -a_1^2 & a_1^1 \end{Vmatrix}. \tag{2.14}$$

Relying on (2.14), we define the map taking a 2×2 matrix to another 2×2 matrix:

$$A \mapsto L(A) = L\left(\begin{Vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{Vmatrix}\right) = \begin{Vmatrix} a_2^2 & -a_2^1 \\ -a_1^2 & a_1^1 \end{Vmatrix}. \tag{2.15}$$

If $\det A = \pm 1$, then for this 2×2 matrix A we have

$$A^{-1} = \det(A) \cdot L(A). \tag{2.16}$$

Note that $\varepsilon_m = \det(\boldsymbol{\sigma}_m) = \pm 1$ for all matrices (2.11). The same is true for the matrix in the left hand side of (2.10) since $\mathfrak{S} \in \mathrm{SL}(2, \mathbb{C})$ and $\det \mathfrak{S} = 1$. Therefore, applying (2.16) to (2.10), we derive

$$(\mathfrak{S}^{-1})^{\dagger} \cdot \boldsymbol{\sigma}_m^{-1} \cdot \mathfrak{S}^{-1} = \varepsilon_m \; L\left(\sum_{k=0}^{3} S^k_m \, \boldsymbol{\sigma}_k\right). \tag{2.17}$$

It is easy to see that the map (2.15) is a linear map. For this reason we can transform the above equality (2.17) in the following way:

$$(\mathfrak{S}^{-1})^{\dagger} \cdot \boldsymbol{\sigma}_m^{-1} \cdot \mathfrak{S}^{-1} = \varepsilon_m \sum_{k=0}^{3} S^k_m \, L(\boldsymbol{\sigma}_k) = \varepsilon_m \sum_{k=0}^{3} S^k_m \; \varepsilon_k \; \sigma_k^{-1}. \tag{2.18}$$

Since $\varepsilon_m = \pm 1$ and $\varepsilon_k = \pm 1$, passing from (2.11) to the matrices (2.13) in (2.18), we see that it coincides with the required equality (2.12). $\square$

The next step is to extend the group $\mathrm{SL}(2, \mathbb{C})$ using the homomorphism (2.9) for this purpose. It is easy to see that the mapping

$$\mathfrak{S} \mapsto \Psi'(\mathfrak{S}) = \hat{\mathfrak{S}} = \left\| \begin{array}{c|c} \mathfrak{S} & 0 \\ \hline 0 & (\mathfrak{S}^{-1})^{\dagger} \end{array} \right\| \tag{2.19}$$

is an embedding of the group $\mathrm{SL}(2, \mathbb{C})$ into the general linear group $\mathrm{GL}(4, \mathbb{C})$, i. e. it is an exact representation of the group $\mathrm{SL}(2, \mathbb{C})$ by means of complex 4×4 matrices. In addition to (2.19), we consider the following 4×4 matrices:

$$\gamma_m = \left\| \begin{array}{c|c} 0 & \boldsymbol{\sigma}_m \\ \hline \tilde{\boldsymbol{\sigma}}_m & 0 \end{array} \right\|, \quad m = 0, 1, 2, 3. \tag{2.20}$$

Then, using the matrices (2.19) and (2.20), we combine them as follows:

$$\hat{\mathfrak{S}} \cdot \gamma_m \cdot \hat{\mathfrak{S}}^{-1} = \left\| \begin{array}{c|c} 0 & \mathfrak{S} \cdot \boldsymbol{\sigma}_m \cdot \mathfrak{S}^{\dagger} \\ \hline (\mathfrak{S}^{-1})^{\dagger} \cdot \tilde{\boldsymbol{\sigma}}_m \cdot \mathfrak{S}^{-1} & 0 \end{array} \right\|. \tag{2.21}$$

Applying (2.10) and (2.18) to (2.21), we can transform this equality to

$$\hat{\mathfrak{S}} \cdot \gamma_m \cdot \hat{\mathfrak{S}}^{-1} = \sum_{k=0}^{3} S^k_m \, \gamma_k. \tag{2.22}$$

The matrices γ_m in (2.20) are known as Dirac matrices. They obey the following anticommutation relationship (see [2]):

$$\{\gamma_i, \, \gamma_j\} = 2 \, g_{ij} \; \mathbf{1}. \tag{2.23}$$

Here $\{\gamma_i, \gamma_j\} = \gamma_i \cdot \gamma_j - \gamma_j \cdot \gamma_i$ is the matrix anticommutator, while $\mathbf{1}$ in (2.23) is the unit 4×4 matrix and g_{ij} are the components of the matrix (1.6), i. e. they are numbers. Due to (2.23) the following combinations of γ-matrices are independent:

$$\mathbf{1} = \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{Vmatrix}, \qquad \gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3 = \begin{Vmatrix} i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & -i \end{Vmatrix}, \tag{2.24}$$

$$\gamma_0 = \begin{Vmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{Vmatrix}, \qquad \gamma_1\cdot\gamma_2\cdot\gamma_3 = \begin{Vmatrix} 0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\\ i & 0 & 0 & 0\\ 0 & i & 0 & 0 \end{Vmatrix}, \tag{2.25}$$

$$\gamma_1 = \begin{Vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{Vmatrix}, \qquad \gamma_0\cdot\gamma_2\cdot\gamma_3 = \begin{Vmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0\\ 0 & -i & 0 & 0\\ -i & 0 & 0 & 0 \end{Vmatrix}, \tag{2.26}$$

$$\gamma_2 = \begin{Vmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0 \end{Vmatrix}, \qquad \gamma_0\cdot\gamma_1\cdot\gamma_3 = \begin{Vmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{Vmatrix}, \tag{2.27}$$

$$\gamma_3 = \begin{Vmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{Vmatrix}, \qquad \gamma_0\cdot\gamma_1\cdot\gamma_2 = \begin{Vmatrix} 0 & 0 & -i & 0\\ 0 & 0 & 0 & i\\ -i & 0 & 0 & 0\\ 0 & i & 0 & 0 \end{Vmatrix}, \tag{2.28}$$

$$\gamma_0\cdot\gamma_1 = \begin{Vmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{Vmatrix}, \qquad \gamma_2\cdot\gamma_3 = \begin{Vmatrix} 0 & -i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & -i & 0 \end{Vmatrix}, \tag{2.29}$$

$$\gamma_0\cdot\gamma_2 = \begin{Vmatrix} 0 & i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0 \end{Vmatrix}, \qquad \gamma_1\cdot\gamma_3 = \begin{Vmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{Vmatrix}, \tag{2.30}$$

$$\gamma_0\cdot\gamma_3 = \begin{Vmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{Vmatrix}, \qquad \gamma_1\cdot\gamma_2 = \begin{Vmatrix} -i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & i \end{Vmatrix}, \tag{2.31}$$

All other products of Dirac matrices are expressed as linear combinations of these matrices. Moreover, the matrices (2.24), (2.25), (2.26), (2.27), (2.28), (2.29), (2.30), (2.31) are linearly independent over the field of complex numbers $\mathbb{C}$. They form a basis in the linear space of 4×4 complex matrices. This fact is well-known, it is mentioned in [2].

In order to extend the group $\mathrm{SL}(2,\mathbb{C})$ represented as a subgroup $G\subset \mathrm{GL}(4,\mathbb{C})$ by means of the embedding (2.19) we modify the relationship (2.22) as follows. We replace S^k_m in (2.22) by the components of the spatial inversion matrix P from (2.2). Then we replace $\hat{\mathfrak{S}}$ by some 4×4 matrix $\hat{P}$ which is yet unknown:

$$\hat{P}\cdot\gamma_m\cdot\hat{P}^{-1} = \sum_{k=0}^{3} P^k_m\,\gamma_k. \tag{2.32}$$

Our next goal is to solve (2.32) with respect to the unknown matrix $\hat{P}$. Note that it can be written as a system of four very simple matrix equations:

$$\hat{P}\cdot\gamma_0=\gamma_0\cdot\hat{P}, \qquad \hat{P}\cdot\gamma_1=-\gamma_1\cdot\hat{P},$$
$$\hat{P}\cdot\gamma_2=-\gamma_2\cdot\hat{P}, \qquad \hat{P}\cdot\gamma_3=-\gamma_3\cdot\hat{P}.$$

This is the system of 64 linear homogeneous equations with respect to 16 components of the matrix $\hat{P}$. Its general solution is given by the formula

$$\hat{P}=C\,\gamma_0, \tag{2.33}$$

where C is an arbitrary complex number.

In a similar way, taking the components of time inversion matrix T from (2.2), on the base of (2.22) we can write the equation

$$\hat{T}\cdot\gamma_m\cdot\hat{T}^{-1}=\sum_{k=0}^{3}T^k_m\,\gamma_k \tag{2.34}$$

for the unknown matrix $\hat{T}$. Like the equation (2.32), the equation (2.34) reduces to a system of four matrix equations:

$$\hat{T}\cdot\gamma_0=-\gamma_0\cdot\hat{T}, \qquad \hat{T}\cdot\gamma_1=\gamma_1\cdot\hat{T},$$
$$\hat{T}\cdot\gamma_2=\gamma_2\cdot\hat{T}, \qquad \hat{T}\cdot\gamma_3=\gamma_3\cdot\hat{T}.$$

The general solution of this system of equations is given by the formula

$$T=C\,\gamma_1\cdot\gamma_2\cdot\gamma_3, \tag{2.35}$$

where C again is an arbitrary complex number. In order to fix the constants in (2.33) and (2.35) we apply the following restrictions[1] to $\hat{P}$ and $\hat{T}$:

$$\hat{P}^2=\mathbf{1}, \qquad \hat{T}^2=\mathbf{1}. \tag{2.36}$$

From (2.36), (2.33), and (2.35) we derive

$$\hat{P}=\pm\,\gamma_0, \qquad \hat{T}=\pm\,\gamma_1\cdot\gamma_2\cdot\gamma_3. \tag{2.37}$$

Moreover, from (2.36) and (2.37) we derive

$$\{\hat{P},\hat{T}\}=0, \qquad (\hat{P}\cdot\hat{T})^2=-\mathbf{1}. \tag{2.38}$$

The first equality (2.38) means that $\hat{P}$ and $\hat{T}$ are anticommutative with respect to each other. The relationships (2.38) are valid for any choice of sign in (2.37). This fact can be strengthened in the following way.

Lemma 2.2. *For any choice of signs the matrices* (2.37) *and the matrices* (2.19), *where* $\mathfrak{S}\in\mathrm{SL}(2,\mathbb{C})$, *generate the same subgroup* $G\subset\mathrm{GL}(4,\mathbb{C})$ *being a discrete extension of the group* $\mathrm{SL}(2,\mathbb{C})$.

[1] In some cases other normalization conditions for $\hat{P}$ and $\hat{T}$ are used, see [7] and [8].

The group G in the lemma 2.2 is isomorphic to the spinor group $\mathrm{Pin}(1,3,\mathbb{R})$. If we identify G with $\mathrm{Pin}(1,3,\mathbb{R})$ according to this isomorphism, then the subgroup[1] $\mathrm{Spin}(1,3,\mathbb{R}) \subset \mathrm{Pin}(1,3,\mathbb{R})$ is identified with the 4-dimensional presentation of the group $\mathrm{SL}(2,\mathbb{C})$ given by the matrices (2.19). Thus we have reached the goal stated in the very beginning of this section. By introducing the matrices (2.19) and (2.37) we have constructed the group homomorphism

$$\Phi\colon\ \mathrm{Pin}(1,3,\mathbb{R}) \to \mathrm{O}(1,3,\mathbb{R}). \tag{2.39}$$

This homomorphism (2.39) extends the initial homomorphism (1.1) in the sense of the following commutative diagram:

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{C}) & \xrightarrow{\ \varphi\ } & \mathrm{SO}(1,3,\mathbb{R}) \\ {\scriptstyle\Psi'}\big\downarrow & & \big\downarrow \\ \mathrm{Pin}(1,3,\mathbb{R}) \cong G & \xrightarrow{\ \Phi\ } & \mathrm{O}(1,3,\mathbb{R}). \end{array} \tag{2.40}$$

Both vertical arrows in the diagram (2.40) are embeddings. The homomorphism (2.39) is described by the formula

$$\hat{\mathfrak{S}} \cdot \gamma_m \cdot \hat{\mathfrak{S}}^{-1} = \sum_{k=0}^{3} S^k_m\,\gamma_k. \tag{2.41}$$

This formula coincides with (2.22), however, now $\hat{\mathfrak{S}}$ is not necessarily given by the formula (2.19). It is an arbitrary matrix from the group $G \cong \mathrm{Pin}(1,3,\mathbb{R})$. In particular, we can take $\hat{\mathfrak{S}} = \hat{P}$ or $\hat{\mathfrak{S}} = \hat{T}$. Then (2.41) reduces to (2.32) or to (2.34) respectively. By S^k_m now in the formula (2.41) we denote the components of the Lorentzian matrix $S = \Phi(\hat{\mathfrak{S}}) \in \mathrm{O}(1,3,\mathbb{R})$.

Like (1.1), the homomorphism (2.39) is a srjective mapping. Its kernel is discrete, it is composed by the following two matrices:

$$\mathbf{1} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \qquad\qquad -\,\mathbf{1} = \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{2.42}$$

Due to (2.42) for any matrix $S \in \mathrm{O}(1,3,\mathbb{R})$ its preimage $\hat{\mathfrak{S}} \in G \cong \mathrm{Pin}(1,3,\mathbb{R})$ is determined uniquely up to the sign.

3. Dirac spinors.

Let M be a space-time manifold and let SM be a spinor bundle over M introduced by the definition 1.1. By $S^\dagger M$ we denote the Hermitian conjugate bundle for SM. Taking both SM and $S^\dagger M$, we construct their direct sum

$$DM = SM \oplus S^\dagger M. \tag{3.1}$$

[1] The definitions of the groups $\mathrm{Spin}(1,3,\mathbb{R})$ and $\mathrm{Pin}(1,3,\mathbb{R})$ can be found in [9].

The direct sum (3.1) is called the *Dirac bundle* associated with the spinor bundle SM. This is a four-dimensional complex bundle over M. The bundles SM and $S^{\dagger}M$, when treated as the constituents of DM, are called *chiral bundles*. Local and global smooth sections of the Dirac bundle DM are called *spinor fields* or more precisely *spinor fields of Dirac spinors*.

According to the definition 1.1, the chiral bundle SM in (3.1) is equipped with the spin-metric $\mathbf{d}$. This metric induces dual metric $\mathbf{d}$ in $S^{*}M$. Then by means of the semilinear isomorphism of complex conjugation

$$\tau\colon\ S^{*}M \to S^{\dagger}M \tag{3.2}$$

it is transferred to the Hermitian conjugate bundle $S^{\dagger}M$ (see more details in [4]). Having spin-metrics in SM and in $S^{\dagger}M$, we can define a spin-metric in DM. Indeed, let $\mathbf{X}\in D_p(M)$ and $\mathbf{Y}\in D_p(M)$ at some point $p\in M$. Then due to (3.1) we have

$$\begin{aligned}&\mathbf{X}=\mathbf{X}_1+\mathbf{X}_2, \ \text{ where }\ \mathbf{X}_1\in S_p(M)\ \text{ and }\ \mathbf{X}_2\in S^{\dagger}_p(M),\\ &\mathbf{Y}=\mathbf{Y}_1+\mathbf{Y}_2, \ \text{ where }\ \mathbf{Y}_1\in S_p(M)\ \text{ and }\ \mathbf{Y}_2\in S^{\dagger}_p(M).\end{aligned} \tag{3.3}$$

Using the expansions (3.3), by definition we set

$$d(\mathbf{X},\mathbf{Y})=d(\mathbf{X}_1,\mathbf{Y}_1)+d(\mathbf{X}_2,\mathbf{Y}_2). \tag{3.4}$$

Thus, the skew-symmetric metric (3.4) in DM is introduced as the sum of metrics in SM and $S^{\dagger}M$ due to the expansion (3.1).

For each vector $\mathbf{X}\in D_p(M)$ we have the expansion (3.3) determined by the expansion (3.1). The operator $\mathbf{H}$ then is defined by the formula

$$\mathbf{H}(\mathbf{X})=\mathbf{X}_1-\mathbf{X}_2. \tag{3.5}$$

This formula means that $\mathbf{H}$ in each fiber $D_p(M)$ is defined as a linear operator with two eigenvalues $\lambda_1=1$ and $\lambda_2=-1$. The eigenspace for λ_1 coincides with $S_p(M)$ and the eigenspace for λ_2 coincides with $S^{\dagger}_p(M)$. The operator field $\mathbf{H}$ introduced by the formula (3.5) is called the *chirality operator*.

The inverse map for (3.2) is denoted by the same symbol τ. It is also called the semilinear isomorphism of complex conjugation (see [4]):

$$\tau\colon\ S^{\dagger}M \to S^{*}M. \tag{3.6}$$

Applying (3.6) to $\mathbf{X}_2$ and $\mathbf{Y}_2$ in (3.3), we get two chiral cospinors in $S^{*}_p(M)$:

$$\mathbf{x}_2=\tau(\mathbf{X}_2), \qquad\qquad \mathbf{y}_2=\tau(\mathbf{Y}_2). \tag{3.7}$$

Since $\mathbf{x}_2\in S^{*}_p(M)$ and $\mathbf{x}_2\in S^{*}_p(M)$, they can be paired with $\mathbf{X}_1$ and $\mathbf{Y}_1$. Hence, we can define the following pairing for the spinors $\mathbf{X}$ and $\mathbf{Y}$:

$$D(\mathbf{X},\mathbf{Y})=(\mathbf{x}_2,\mathbf{Y}_1)+\overline{(\mathbf{y}_2,\mathbf{X}_1)} \tag{3.8}$$

The Hermitian form $\mathbf{D}$ defined by means of the formulas (3.7) and (3.8) is called the *Dirac form* or the *Hermitian spin-metric*. Note that the Hermitian spin-metric $\mathbf{D}$ is not positive, its signature is $(+,+,-,-)$.

The spin-metric $\mathbf{d}$, the chirality operator $\mathbf{H}$ and the Hermitian spin-metric $\mathbf{D}$ are basic geometric structures associated with Dirac spinors. Some other structures will be considered below a little bit later.

4. Spin-tensors.

The definition of spin-tensors in the case of Dirac spinors is quite standard. We introduce them following the scheme of the paper [4]. Let $T_p(M)$ and $D_p(M)$ be the fibers of the tangent bundle TM and the Dirac bundle DM at some point $p \in M$. Denote by $T^*_p(M)$ and $D^*_p(M)$ the dual spaces for $T_p(M)$ and $D_p(M)$, then produce from $T_p(M)$ and $T^*_p(M)$ the complex spaces $\mathbb{C}T_p(M)$ and $\mathbb{C}T^*_p(M)$ by means of standard complexification procedure:

$$\mathbb{C}T_p(M) = \mathbb{C} \otimes T_p(M), \qquad\qquad \mathbb{C}T^*_p(M) = \mathbb{C} \otimes T^*_p(M). \tag{4.1}$$

The complex spaces (4.1) are obviously dual to each other. In addition to $D_p(M)$ and $D^*_p(M)$ we introduce the Hermitian conjugate spaces

$$D^\dagger_p(M), \qquad\qquad D^{*\dagger}_p(M) = D^{\dagger *}_p(M). \tag{4.2}$$

Then, using (4.1) and (4.2), we define the following tensor products:

$$\mathbb{C}T^m_n(p,M) = \overbrace{\mathbb{C}T_p(M) \otimes \ldots \otimes \mathbb{C}T_p(M)}^{m \text{ times}} \otimes \underbrace{\mathbb{C}T^*_p(M) \otimes \ldots \otimes \mathbb{C}T^*_p(M)}_{n \text{ times}}, \tag{4.3}$$

$$D^\alpha_\beta(p,M) = \overbrace{D_p(M) \otimes \ldots \otimes D_p(M)}^{\alpha \text{ times}} \otimes \underbrace{D^*_p(M) \otimes \ldots \otimes D^*_p(M)}_{\beta \text{ times}}, \tag{4.4}$$

$$\bar{D}^\nu_\gamma(p,M) = \overbrace{D^{\dagger *}_p(M) \otimes \ldots \otimes D^{\dagger *}_p(M)}^{\nu \text{ times}} \otimes \underbrace{D^\dagger_p(M) \otimes \ldots \otimes D^\dagger_p(M)}_{\gamma \text{ times}}. \tag{4.5}$$

Combining (4.3), (4.4), and (4.5), we define one more tensor product

$$D^\alpha_\beta \bar{D}^\nu_\gamma T^r_s(p,M) = D^\alpha_\beta(p,M) \otimes \bar{D}^\nu_\gamma(p,M) \otimes \mathbb{C}T^m_n(p,M). \tag{4.6}$$

Elements of the space (4.6) are called *Dirac spin-tensors* of the type $(\alpha,\beta|\nu,\gamma|m,n)$ at the point $p \in M$. The spaces (4.6) with p running over the space-time manifold M are naturally glued into a bundle. This bundle is called the *spin-tensorial bundle* of the type $(\alpha,\beta|\nu,\gamma|m,n)$, its local and global smooth sections are called *spin-tensorial fields* of the type $(\alpha,\beta|\nu,\gamma|m,n)$.

The complex conjugation isomorphism τ for Dirac spin-tensors is introduced in the same way as in the case of chiral spin-tensors. The tangent space $T_p(M)$ and the cotangent space $T^*_p(M)$ are real spaces. Therefore, here we have

$$\begin{aligned} &\tau(\mathbf{X}) = \mathbf{X} \ \text{ for all } \ \mathbf{X} \in T_p(M),\\ &\tau(\lambda\,\mathbf{X}) = \overline{\lambda}\,\mathbf{X} \ \text{ for any } \ \lambda \in \mathbb{C}. \end{aligned} \tag{4.7}$$

Similarly, in the case of the cotangent space $T^*_p(M)$ we have

$$\begin{aligned}&\tau(\mathbf{u})=\mathbf{u}\ \text{ for all }\ \mathbf{u}\in T^*_p(M),\\&\tau(\lambda\,\mathbf{u})=\overline{\lambda}\,\mathbf{u}\ \text{ for any }\ \lambda\in\mathbb{C}.\end{aligned}\tag{4.8}$$

The formulas (4.7) and (4.8) define τ as two semilinear mappings

$$\tau\colon\ \mathbb{C}T_p(M)\to\mathbb{C}T_p(M),\qquad\qquad \tau\colon\ \mathbb{C}T^*_p(M)\to\mathbb{C}T^*_p(M)\tag{4.9}$$

such that $\tau^2=\tau\circ\tau=\mathbf{id}$. The mappings (4.9) are easily extended to the tensor product (4.3). As a result we have the semilinear mapping

$$\tau\colon\ \mathbb{C}T^m_n(p,M)\to\mathbb{C}T^m_n(p,M)\tag{4.10}$$

with the same property $\tau^2=\tau\circ\tau=\mathbf{id}$. Apart from the mappings (4.9), we have the following canonical semilinear mappings mutually inverse in each pair:

$$D_p(M)\ \underset{\tau}{\overset{\tau}{\rightleftarrows}}\ D^{*\dagger}_p(M),\qquad\qquad D^*_p(M)\ \underset{\tau}{\overset{\tau}{\rightleftarrows}}\ D^{\dagger}_p(M).\tag{4.11}$$

They are defined according to the recipe of the section 3 in [4]. All of the mappings (4.11) are denoted by the same symbol τ so that we formally preserve the property $\tau^2=\tau\circ\tau=\mathbf{id}$. They are easily extended to the tensor products (4.4) and (4.5):

$$D^\alpha_\beta(p,M)\ \underset{\tau}{\overset{\tau}{\rightleftarrows}}\ \bar{D}^\alpha_\beta(p,M),\tag{4.12}$$

Note that the second pair of the mappings (4.11) are similar to (3.2) and (3.6), though here they are defined independently. Now, combining the mappings (4.10) and (4.12), we extend τ to the tensor product (4.6):

$$D^\alpha_\beta\bar{D}^\nu_\gamma T^r_s(p,M)\ \underset{\tau}{\overset{\tau}{\rightleftarrows}}\ D^\nu_\gamma\bar{D}^\alpha_\beta T^r_s(p,M).\tag{4.13}$$

The mappings (4.13) are inverse to each other so that the property $\tau^2=\tau\circ\tau=\mathbf{id}$ for them is again formally preserved.

Let $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ and $(\tilde{U},\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2,\,\tilde{\mathbf{\Psi}}_3,\,\tilde{\mathbf{\Psi}}_4)$ be two frames of the Dirac bundle DM with overlapping domains: $U\cap\tilde{U}\neq\varnothing$. Let $(U,\,\mathbf{\Upsilon}_0,\,\mathbf{\Upsilon}_1,\,\mathbf{\Upsilon}_2,\,\mathbf{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\mathbf{\Upsilon}}_0,\,\tilde{\mathbf{\Upsilon}}_1,\,\tilde{\mathbf{\Upsilon}}_2,\,\tilde{\mathbf{\Upsilon}}_3)$ be two frames of the tangent bundle TM with the same domains U and $\tilde{U}$. Note that here we do not require $(U,\,\mathbf{\Upsilon}_0,\,\mathbf{\Upsilon}_1,\,\mathbf{\Upsilon}_2,\,\mathbf{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\mathbf{\Upsilon}}_0,\,\tilde{\mathbf{\Upsilon}}_1,\,\tilde{\mathbf{\Upsilon}}_2,\,\tilde{\mathbf{\Upsilon}}_3)$ to be positively polarized right orthonormal frames. Despite to this higher level of arbitrariness, here we can write the relationships (1.7) and the following relationships for spinor frames:

$$\tilde{\mathbf{\Psi}}_i=\sum^4_{j=1}\hat{\mathfrak{S}}^j_i\ \mathbf{\Psi}_j,\qquad\qquad \mathbf{\Psi}_i=\sum^4_{j=1}\hat{\mathfrak{T}}^j_i\ \tilde{\mathbf{\Psi}}_j.\tag{4.14}$$

Now S^j_i and $\hat{\mathfrak{S}}^j_i$ in (1.7) and (4.14) are the components of arbitrary two 4×4 matrices, while T^j_i and $\hat{\mathfrak{T}}^j_i$ are the components of their inverse matrices. Let's denote by $(U,\,\boldsymbol{\vartheta}^{\,1},\,\boldsymbol{\vartheta}^{\,2},\,\boldsymbol{\vartheta}^{\,3},\,\boldsymbol{\vartheta}^{\,4})$ the dual frame for $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2,\,\boldsymbol{\Psi}_3,\,\boldsymbol{\Psi}_4)$ and denote by $(\tilde{U},\,\tilde{\boldsymbol{\vartheta}}^{\,1},\,\tilde{\boldsymbol{\vartheta}}^{\,2},\,\tilde{\boldsymbol{\vartheta}}^{\,3},\,\tilde{\boldsymbol{\vartheta}}^{\,4})$ the dual frame for $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2,\,\tilde{\boldsymbol{\Psi}}_3,\,\tilde{\boldsymbol{\Psi}}_4)$. Then

$$\tilde{\boldsymbol{\vartheta}}^{\,i}=\sum_{j=1}^{4}\hat{\mathfrak{T}}^i_j\;\boldsymbol{\vartheta}^{\,j},\qquad\qquad \boldsymbol{\vartheta}^{\,i}=\sum_{j=1}^{4}\hat{\mathfrak{S}}^i_j\;\tilde{\boldsymbol{\vartheta}}^{\,j}. \tag{4.15}$$

Applying τ to $\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2,\,\boldsymbol{\Psi}_3,\,\boldsymbol{\Psi}_4,\,\boldsymbol{\vartheta}^{\,1},\,\boldsymbol{\vartheta}^{\,2},\,\boldsymbol{\vartheta}^{\,3},\,\boldsymbol{\vartheta}^{\,4},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2,\,\tilde{\boldsymbol{\Psi}}_3,\,\tilde{\boldsymbol{\Psi}}_4,\,\tilde{\boldsymbol{\vartheta}}^{\,1},\,\tilde{\boldsymbol{\vartheta}}^{\,2},\,\tilde{\boldsymbol{\vartheta}}^{\,3}$, and $\tilde{\boldsymbol{\vartheta}}^{\,4}$, we get four frames in $D^{*\dagger}_p(M)$ and $D^{\dagger}_p(M)$:

$$\overline{\boldsymbol{\Psi}}_i=\tau(\boldsymbol{\Psi}_i),\qquad\qquad \overline{\boldsymbol{\vartheta}}^{\,i}=\tau(\boldsymbol{\vartheta}^{\,i}), \tag{4.16}$$

$$\overline{\tilde{\boldsymbol{\Psi}}}_i=\tau(\tilde{\boldsymbol{\Psi}}_i),\qquad\qquad \overline{\tilde{\boldsymbol{\vartheta}}}^{\,i}=\tau(\tilde{\boldsymbol{\vartheta}}^{\,i}). \tag{4.17}$$

The frames $(U,\,\overline{\boldsymbol{\Psi}}_1,\,\overline{\boldsymbol{\Psi}}_2,\,\overline{\boldsymbol{\Psi}}_3,\,\overline{\boldsymbol{\Psi}}_4)$, $(\tilde{U},\,\overline{\tilde{\boldsymbol{\Psi}}}_1,\,\overline{\tilde{\boldsymbol{\Psi}}}_2,\,\overline{\tilde{\boldsymbol{\Psi}}}_3,\,\overline{\tilde{\boldsymbol{\Psi}}}_4)$, $(U,\,\overline{\boldsymbol{\vartheta}}^{\,1},\,\overline{\boldsymbol{\vartheta}}^{\,2},\,\overline{\boldsymbol{\vartheta}}^{\,3},\,\overline{\boldsymbol{\vartheta}}^{\,4})$, and $(\tilde{U},\,\overline{\tilde{\boldsymbol{\vartheta}}}^{\,1},\,\overline{\tilde{\boldsymbol{\vartheta}}}^{\,2},\,\overline{\tilde{\boldsymbol{\vartheta}}}^{\,3},\,\overline{\tilde{\boldsymbol{\vartheta}}}^{\,4})$ are related to each other as follows:

$$\overline{\tilde{\boldsymbol{\Psi}}}_i=\sum_{j=1}^{4}\overline{\hat{\mathfrak{S}}^j_i}\;\overline{\boldsymbol{\Psi}}_j,\qquad\qquad \overline{\boldsymbol{\Psi}}_i=\sum_{j=1}^{4}\overline{\hat{\mathfrak{T}}^j_i}\;\overline{\tilde{\boldsymbol{\Psi}}}_j, \tag{4.18}$$

$$\overline{\tilde{\boldsymbol{\vartheta}}}^{\,i}=\sum_{j=1}^{4}\overline{\hat{\mathfrak{T}}^i_j}\;\overline{\boldsymbol{\vartheta}}^{\,j},\qquad\qquad \overline{\boldsymbol{\vartheta}}^{\,i}=\sum_{j=1}^{4}\overline{\hat{\mathfrak{S}}^i_j}\;\overline{\tilde{\boldsymbol{\vartheta}}}^{\,j}. \tag{4.19}$$

The formulas (4.18) and (4.19) are derived from (4.14) and (4.15) by applying the relationships (4.16) and (4.17). And finally, we have the relationships

$$\tilde{\boldsymbol{\eta}}^i=\sum_{j=0}^{3}T^i_j\;\boldsymbol{\eta}^j,\qquad\qquad \boldsymbol{\eta}^i=\sum_{j=0}^{3}S^i_j\;\tilde{\boldsymbol{\eta}}^j, \tag{4.20}$$

where $(U,\,\boldsymbol{\eta}^0,\,\boldsymbol{\eta}^1,\,\boldsymbol{\eta}^2,\,\boldsymbol{\eta}^3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\eta}}^0,\,\tilde{\boldsymbol{\eta}}^1,\,\tilde{\boldsymbol{\eta}}^2,\,\tilde{\boldsymbol{\eta}}^3)$ are the frames dual to the frames $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ respectively.

Let's use the above frame vectors, covectors, spinors, and cospinors (including complex conjugate ones) in order to introduce the following tensor products:

$$\begin{aligned}\boldsymbol{\Upsilon}^{k_1\ldots\,k_n}_{h_1\ldots\,h_m}&=\boldsymbol{\Upsilon}_{h_1}\otimes\ldots\otimes\boldsymbol{\Upsilon}_{h_m}\otimes\boldsymbol{\eta}^{k_1}\otimes\ldots\otimes\boldsymbol{\eta}^{k_n},\\ \tilde{\boldsymbol{\Upsilon}}^{k_1\ldots\,k_n}_{h_1\ldots\,h_m}&=\tilde{\boldsymbol{\Upsilon}}_{h_1}\otimes\ldots\otimes\tilde{\boldsymbol{\Upsilon}}_{h_m}\otimes\tilde{\boldsymbol{\eta}}^{k_1}\otimes\ldots\otimes\tilde{\boldsymbol{\eta}}^{k_n},\end{aligned} \tag{4.21}$$

$$\begin{aligned}\boldsymbol{\Psi}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}&=\boldsymbol{\Psi}_{i_1}\otimes\ldots\otimes\boldsymbol{\Psi}_{i_\alpha}\otimes\boldsymbol{\vartheta}^{\,j_1}\otimes\ldots\otimes\boldsymbol{\vartheta}^{\,j_\beta},\\ \tilde{\boldsymbol{\Psi}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}&=\tilde{\boldsymbol{\Psi}}_{i_1}\otimes\ldots\otimes\tilde{\boldsymbol{\Psi}}_{i_\alpha}\otimes\tilde{\boldsymbol{\vartheta}}^{\,j_1}\otimes\ldots\otimes\tilde{\boldsymbol{\vartheta}}^{\,j_\beta},\end{aligned} \tag{4.22}$$

$$\begin{aligned}\overline{\boldsymbol{\Psi}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}&=\overline{\boldsymbol{\Psi}}_{\bar{i}_1}\otimes\ldots\otimes\overline{\boldsymbol{\Psi}}_{\bar{i}_\nu}\otimes\overline{\boldsymbol{\vartheta}}^{\,j_1}\otimes\ldots\otimes\overline{\boldsymbol{\vartheta}}^{\,j_\gamma},\\ \overline{\tilde{\boldsymbol{\Psi}}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}&=\overline{\tilde{\boldsymbol{\Psi}}}_{\bar{i}_1}\otimes\ldots\otimes\overline{\tilde{\boldsymbol{\Psi}}}_{\bar{i}_\nu}\otimes\overline{\tilde{\boldsymbol{\vartheta}}}^{\,j_1}\otimes\ldots\otimes\overline{\tilde{\boldsymbol{\vartheta}}}^{\,j_\gamma}.\end{aligned} \tag{4.23}$$

Then, using (4.21), (4.22), and (4.23), we define other two tensor products:

$$\boldsymbol{\Psi}^{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}=\boldsymbol{\Psi}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}\otimes\overline{\boldsymbol{\Psi}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}\otimes\boldsymbol{\Upsilon}^{k_1\ldots\,k_n}_{h_1\ldots\,h_m}, \tag{4.24}$$

$$\tilde{\mathbf{\Psi}}^{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}=\tilde{\mathbf{\Psi}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}\otimes\overline{\tilde{\mathbf{\Psi}}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}\otimes\tilde{\mathbf{\Upsilon}}^{k_1\ldots\,k_n}_{h_1\ldots\,h_m}. \tag{4.25}$$

Both tensor products (4.24) and (4.25) are spin-tensorial fields of the same type $(\alpha,\beta|\nu,\gamma|m,n)$. They are used in order to expand other spin-tensorial field of this type. If $\mathbf{X}$ is a spin-tensorial field of the type $(\alpha,\beta|\nu,\gamma|m,n)$, then

$$\mathbf{X}=\sum^{4}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta}}\!\!\ldots\!\sum^{4}\sum^{4}_{\substack{\bar{i}_1,\,\ldots\,,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\gamma}}\!\!\ldots\!\sum^{4}\sum^{3}_{\substack{h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\!\sum^{3}X^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}\ \mathbf{\Psi}^{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}. \tag{4.26}$$

The coefficients $X^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}$ in (4.26) are called the *component* of the field $\mathbf{X}$ in the pair of frames $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ and $(U,\,\mathbf{\Upsilon}_0,\,\mathbf{\Upsilon}_1,\,\mathbf{\Upsilon}_2,\,\mathbf{\Upsilon}_3)$. Similarly, the coefficients $\tilde{X}^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}$ in the expansion

$$\mathbf{X}=\sum^{4}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta}}\!\!\ldots\!\sum^{4}\sum^{4}_{\substack{\bar{i}_1,\,\ldots\,,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\gamma}}\!\!\ldots\!\sum^{4}\sum^{3}_{\substack{h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\!\sum^{3}\tilde{X}^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}\ \tilde{\mathbf{\Psi}}^{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m} \tag{4.27}$$

are called the components of the field $\mathbf{X}$ in the pair of frames $(\tilde{U},\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2,\,\tilde{\mathbf{\Psi}}_3,\,\tilde{\mathbf{\Psi}}_4)$ and $(\tilde{U},\,\tilde{\boldsymbol{\eta}}^0,\,\tilde{\boldsymbol{\eta}}^1,\,\tilde{\boldsymbol{\eta}}^2,\,\tilde{\boldsymbol{\eta}}^3)$. Applying (1.7), (4.14), (4.15), (4.18), (4.19), and (4.20) to (4.26) and (4.27), we derive the following relationships

$$\begin{gathered}\tilde{X}^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}=\sum^{4}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta}}\!\!\ldots\!\sum^{4}\sum^{4}_{\substack{\bar{a}_1,\,\ldots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\gamma}}\!\!\ldots\!\sum^{4}\sum^{3}_{\substack{c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\!\!\ldots\!\sum^{3}\hat{\mathfrak{T}}^{i_1}_{a_1}\ldots\,\hat{\mathfrak{T}}^{i_\alpha}_{a_\alpha}\,\times\\ \times\,\hat{\mathfrak{S}}^{b_1}_{j_1}\ldots\,\hat{\mathfrak{S}}^{b_\beta}_{j_\beta}\,\overline{\hat{\mathfrak{T}}^{\bar{i}_1}_{\bar{a}_1}}\ldots\,\overline{\hat{\mathfrak{T}}^{\bar{i}_\nu}_{\bar{a}_\nu}}\ \overline{\hat{\mathfrak{S}}^{\bar{b}_1}_{\bar{j}_1}}\ldots\,\overline{\hat{\mathfrak{S}}^{\bar{b}_\gamma}_{\bar{j}_\gamma}}\ T^{h_1}_{c_1}\ldots\,T^{h_m}_{c_m}\,\times\\ \times\,S^{d_1}_{k_1}\ldots\,S^{d_n}_{k_n}\ X^{a_1\ldots\,a_\alpha\,\bar{a}_1\ldots\,\bar{a}_\nu\,c_1\ldots\,c_m}_{b_1\ldots\,b_\beta\,\bar{b}_1\ldots\,\bar{b}_\gamma\,d_1\ldots\,d_n},\end{gathered} \tag{4.28}$$

$$\begin{gathered}X^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}=\sum^{4}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta}}\!\!\ldots\!\sum^{4}\sum^{4}_{\substack{\bar{a}_1,\,\ldots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\gamma}}\!\!\ldots\!\sum^{4}\sum^{3}_{\substack{c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\!\!\ldots\!\sum^{3}\hat{\mathfrak{S}}^{i_1}_{a_1}\ldots\,\hat{\mathfrak{S}}^{i_\alpha}_{a_\alpha}\,\times\\ \times\,\hat{\mathfrak{T}}^{b_1}_{j_1}\ldots\,\hat{\mathfrak{T}}^{b_\beta}_{j_\beta}\,\overline{\hat{\mathfrak{S}}^{\bar{i}_1}_{\bar{a}_1}}\ldots\,\overline{\hat{\mathfrak{S}}^{\bar{i}_\nu}_{\bar{a}_\nu}}\ \overline{\hat{\mathfrak{T}}^{\bar{b}_1}_{\bar{j}_1}}\ldots\,\overline{\hat{\mathfrak{T}}^{\bar{b}_\gamma}_{\bar{j}_\gamma}}\ S^{h_1}_{c_1}\ldots\,S^{h_m}_{c_m}\,\times\\ \times\,T^{d_1}_{k_1}\ldots\,T^{d_n}_{k_n}\ \tilde{X}^{a_1\ldots\,a_\alpha\,\bar{a}_1\ldots\,\bar{a}_\nu\,c_1\ldots\,c_m}_{b_1\ldots\,b_\beta\,\bar{b}_1\ldots\,\bar{b}_\gamma\,d_1\ldots\,d_n}.\end{gathered} \tag{4.29}$$

The formulas (4.28) and (4.29) represent the general transformation rule for the components of spin-tensors in the case of Dirac bundle DM. They are inverse to each other. Below we shall see various special cases of them.

Let's return back to the semilinear isomorphism of complex conjugation τ. From (4.7) and (4.8) we derive the following relationships for τ:

$$\tau(\mathbf{\Upsilon}_i)=\mathbf{\Upsilon}_i, \qquad\qquad \tau(\boldsymbol{\eta}^i)=\boldsymbol{\eta}^i. \tag{4.30}$$

Now, if we combine (4.30) with (4.16) and (4.17) and if we remember the identity $\tau^2 = \tau \circ \tau = \mathbf{id}$, then we derive the following formula:

$$\tau(\mathbf{X}) = \sum_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar{i}_1,\,\ldots\,,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\gamma}}^{4}\!\!\ldots\sum^{4}\sum_{\substack{h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}^{3}\!\!\ldots\sum^{3}\overline{X^{\bar{i}_1\ldots\,\bar{i}_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,h_m}_{\bar{j}_1\ldots\,\bar{j}_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,k_n}}\;\mathbf{\Psi}^{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}_{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}. \tag{4.31}$$

The formula (4.31) means that the isomorphism τ acts upon the components of the expansion (4.26) as the complex conjugation exchanging barred and non-barred indices of them.

5. Coordinate representation of the basic spin-tensorial fields.

Let $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ be an orthonormal frame of the chiral bundle SM. Then it induces three other orthonormal frames: $(U,\,\boldsymbol{\vartheta}^1,\,\boldsymbol{\vartheta}^2)$ in S^*M, $(U,\,\overline{\mathbf{\Psi}}_1,\,\overline{\mathbf{\Psi}}_2)$ in $S^{*\dagger}M$, and $(U,\,\overline{\boldsymbol{\vartheta}}^1,\overline{\boldsymbol{\vartheta}}^2)$ in $S^{\dagger}M$. Due to the expansion (3.1) two frames $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ and $(U,\,\overline{\boldsymbol{\vartheta}}^1,\overline{\boldsymbol{\vartheta}}^2)$ compose a frame in DM. Let's denote

$$\mathbf{\Psi}_3 = \overline{\boldsymbol{\vartheta}}^1, \qquad\qquad \mathbf{\Psi}_4 = \overline{\boldsymbol{\vartheta}}^2. \tag{5.1}$$

Definition 5.1. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM produced from some orthonormal frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ of the chiral bundle SM by virtue of the formula (5.1) is called a *canonically orthonormal chiral frame* of DM.

Let's consider the spin-metric tensor $\mathbf{d}$ introduced by the formula (3.4). In a canonically orthonormal chiral frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ it is given by the matrix

$$d_{ij} = d(\mathbf{\Psi}_i, \mathbf{\Psi}_j) = \begin{Vmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{Vmatrix}. \tag{5.2}$$

The matrix (5.2) is a block-diagonal matrix composed of two diagonal blocks. Its upper left diagonal block coincide with the matrix (1.8) and its lower right diagonal block is given by the matrix inverse to (1.8):

$$\bar{d}^{\,ij} = \bar{d}(\overline{\boldsymbol{\vartheta}}^{\,i}, \overline{\boldsymbol{\vartheta}}^{\,j}) = \overline{d(\boldsymbol{\vartheta}^{\,i}, \boldsymbol{\vartheta}^{\,j})} = \begin{Vmatrix} 0 & -1\\ 1 & 0 \end{Vmatrix}.$$

Definition 5.2. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called an *orthonormal frame* if the spin-metric tensor $\mathbf{d}$ is represented by the matrix (5.2) in this frame.

The chirality operator $\mathbf{H}$ is introduced by the formula (3.5). It is easy to see that in a canonically orthonormal chiral frame it is represented by the matrix

$$H^i_j = \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{5.3}$$

Definition 5.3. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called a *chiral frame* if the chirality operator $\mathbf{H}$ is given by the matrix (5.3) in this frame.

The Hermitian spin-metric tensor $\mathbf{D}$ (it is also called the Dirac form) is introduced by the formulas (3.7) and (3.8). In a canonically orthonormal chiral frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ it is given by the matrix

$$D_{i\bar{j}} = D(\mathbf{\Psi}_{\bar{j}}, \mathbf{\Psi}_i) = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \tag{5.4}$$

Definition 5.4. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called a *self-adjoint frame* if the Dirac form $\mathbf{D}$ is given by the matrix (5.4) in this frame.

The following theorem links together the above four definitions.

Theorem 5.1. *A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is a canonically orthonormal chiral frame if and only if it is orthonormal, chiral, and self-adjoint at the same time.*

The theorem 5.1 shows that three basic spin-tensorial fields $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ describe completely the chiral expansion (3.1) of the Dirac bundle DM.

6. Geometrization of the extended group homomorphism.

Assume that we have two canonically orthonormal chiral frames of the Dirac bundle $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ and $(\tilde{U},\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2,\,\tilde{\mathbf{\Psi}}_3,\,\tilde{\mathbf{\Psi}}_4)$ with overlapping domains. They are associated with the orthonormal frames $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ and $(\tilde{U},\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2)$ of the chiral bundle SM, which in turn are associated with two positively polarized right orthonormal frames $(U,\,\mathbf{\Upsilon}_0,\,\mathbf{\Upsilon}_1,\,\mathbf{\Upsilon}_2,\,\mathbf{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\mathbf{\Upsilon}}_0,\,\tilde{\mathbf{\Upsilon}}_1,\,\tilde{\mathbf{\Upsilon}}_2,\,\tilde{\mathbf{\Upsilon}}_3)$ of the tangent bundle TM. Thus we can write the complete set of transition formulas (1.7), (1.9), (4.14), (4.15), (4.18), (4.19), and (4.20). However, the transition matrices $\hat{\mathfrak{S}}$, $\hat{\mathfrak{T}}$, S, and T are not arbitrary 4×4 matrices in this case. All these matrices are determined by the only one 2×2 matrix $\mathfrak{S}\in \mathrm{SL}(2,\mathbb{C})$. This matrix $\mathfrak{S}$ itself and its inverse matrix $\mathfrak{T}=\mathfrak{S}^{-1}$ are explicitly present in the formulas (1.9). The matrices S and $T=S^{-1}$ in (1.7) and (4.20) are produced from $\mathfrak{S}$ and $\mathfrak{T}$ by means of the homomorphism (1.1):

$$S=\varphi(\mathfrak{S}), \qquad\qquad T=\varphi(\mathfrak{T}). \tag{6.1}$$

The matrices $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}=\hat{\mathfrak{S}}^{-1}$ are produced from $\mathfrak{S}$ and $\mathfrak{T}$ in a more explicit way. They are block-diagonal matrices constructed as follows:

$$\hat{\mathfrak{S}} = \left\Vert \begin{array}{c|c} \mathfrak{S} & 0 \\ \hline 0 & \mathfrak{T}^{\dagger} \end{array} \right\Vert, \qquad\qquad \hat{\mathfrak{T}} = \left\Vert \begin{array}{c|c} \mathfrak{T} & 0 \\ \hline 0 & \mathfrak{S}^{\dagger} \end{array} \right\Vert. \tag{6.2}$$

Canonically orthonormal chiral frames of the Dirac bundle DM are naturally associated with positively polarized right orthonormal frames of the tangent bundle TM. Comparing (6.2) with (2.19), we see that transition matrices relating these two types of frames form a group isomorphic to $\mathrm{SL}(2,\mathbb{C})$ and the group $\mathrm{SO}^{+}(1,3,\mathbb{R})$

respectively. The formulas (6.1) and (6.2) mean that canonically orthonormal chiral frames in DM and positively polarized right orthonormal frames in TM provide a geometrization of the upper line in the commutative diagram (2.40).

The coordinate representations of the basic spin-tensorial fields $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ are invariant when we change a canonically orthonormal chiral frame for another such frame. Indeed, we have the relationships

$$d_{ij} = \sum_{k=1}^{4}\sum_{q=1}^{4} \hat{\mathfrak{T}}^k_i\, \hat{\mathfrak{T}}^q_j\, d_{kq}, \tag{6.3}$$

$$H^i_j = \sum_{k=1}^{4}\sum_{q=1}^{4} \hat{\mathfrak{S}}^i_k\, \hat{\mathfrak{T}}^q_j\, H^k_q, \tag{6.4}$$

$$D_{i\bar{j}} = \sum_{k=1}^{4}\sum_{q=1}^{4} \hat{\mathfrak{T}}^k_i\, \overline{\hat{\mathfrak{T}}^{\bar{q}}_{\bar{j}}}\, D_{k\bar{q}}, \tag{6.5}$$

where $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}$ are block-diagonal matrices of the form (6.2), while the components of $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ are given by the matrices (5.2), (5.3), and (5.4) respectively. The formulas (6.3), (6.4), and (6.5) can be verified by direct calculations.

Comparing (6.3), (6.4), and (6.5) with the general formulas (4.28) and (4.29), we see that the spin-metric tensor $\mathbf{d}$ is a spin-tensorial field of the type $(0,2|0,0|0,0)$, the chirality operator $\mathbf{H}$ is a spin-tensorial field of the type $(1,1|0,0|0,0)$, the Hermitian spin-metric $\mathbf{D}$ is a spin-tensorial field of the type $(0,1|0,1|0,0)$.

In order to extend the above geometric interpretation of the upper line of the commutative diagram (2.40) to its lower line we need to use the matrices (2.37) as transition matrices and apply them to some canonically orthonormal chiral frame $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2,\, \mathbf{\Psi}_3,\, \mathbf{\Psi}_4)$. By setting $\hat{\mathfrak{S}} = \hat{P}$ in (4.14) we get

$$\tilde{\mathbf{\Psi}}_1 = \mathbf{\Psi}_3, \qquad \tilde{\mathbf{\Psi}}_2 = \mathbf{\Psi}_4, \qquad \tilde{\mathbf{\Psi}}_3 = \mathbf{\Psi}_1, \qquad \tilde{\mathbf{\Psi}}_4 = \mathbf{\Psi}_2. \tag{6.6}$$

We choose the plus sign in both formulas (2.37) for the sake of certainty. Since $\hat{P}^2 = \mathbf{1}$ (see (2.36)), from $\hat{\mathfrak{S}} = \hat{P}$ we get $\hat{\mathfrak{T}} = \hat{\mathfrak{S}}^{-1} = \hat{P}$. Then we derive

$$\tilde{d}_{ij} = \sum_{k=1}^{4}\sum_{q=1}^{4} \hat{P}^k_i\, \hat{P}^q_j\, d_{kq} = -d_{ij}, \tag{6.7}$$

$$\tilde{H}^i_j = \sum_{k=1}^{4}\sum_{q=1}^{4} \hat{P}^i_k\, \hat{P}^q_j\, H^k_q = -H^i_j, \tag{6.8}$$

$$\tilde{D}_{i\bar{j}} = \sum_{k=1}^{4}\sum_{q=1}^{4} \hat{P}^k_i\, \overline{\hat{P}^{\bar{q}}_{\bar{j}}}\, D_{k\bar{q}} = D_{i\bar{j}}. \tag{6.9}$$

Like (6.3), (6.4), and (6.5), the formulas (6.7), (6.8), and (6.9) are easily derived by direct calculations.

Note that the components of the chirality operator $\mathbf{H}$ change their signs in (6.8). Therefore, the frame constructed by means of the formulas (6.6) is not a chiral frame, it is an *anti-chiral frame*. The components of the spin-metric tensor $\mathbf{d}$ also change their signs. Hence, the frame (6.6) is not an orthonormal frame in the sense of the definition 5.2. It should be called an *anti-orthonormal frame*, though this is not a commonly used term.

Definition 6.1. A frame $(\tilde{U},\ \tilde{\mathbf{\Psi}}_1,\ \tilde{\mathbf{\Psi}}_2,\ \tilde{\mathbf{\Psi}}_3,\ \tilde{\mathbf{\Psi}}_4)$ produced from some canonically orthonormal chiral frame $(U,\ \mathbf{\Psi}_1,\ \mathbf{\Psi}_2,\ \mathbf{\Psi}_3,\ \mathbf{\Psi}_4)$ by means of the formulas (6.6) is called a *P-reverse anti-chiral frame* of the Dirac bundle DM.

Theorem 6.1. *A frame $(\tilde{U},\ \tilde{\mathbf{\Psi}}_1,\ \tilde{\mathbf{\Psi}}_2,\ \tilde{\mathbf{\Psi}}_3,\ \tilde{\mathbf{\Psi}}_4)$ is a P-reverse anti-chiral frame of the Dirac bundle DM if and only if the components of the basic spin-tensorial fields $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ in this frame are given by the formulas*

$$\tilde{d}_{ij}=-d_{ij},\qquad\qquad \tilde{H}^i_j=-H^i_j,\qquad\qquad \tilde{D}_{i\bar{j}}=D_{i\bar{j}},$$

where d_{ij}, H^i_j, $D_{i\bar{j}}$ are taken from the matrices (5.2), (5.3), and (5.4) respectively.

Note that the spacial inversion matrix $\hat{P}$ from (2.37) is associated with the matrix P in (2.2) by means of the formula (2.32). Therefore, each P-reverse anti-chiral frame of the Dirac bundle DM is canonically associated with some positively polarized left orthonormal frame in TM. This association yields a partial geometrization of the group homomorphism (2.39) forming the lower line in the diagram (2.40). In order complete this scheme of geometrization in the next step we consider the time inversion matrix $\hat{T}$ from (2.37). By setting $\hat{\mathfrak{S}}=\hat{T}$ in (4.14) we get

$$\tilde{\mathbf{\Psi}}_1=i\,\mathbf{\Psi}_3,\qquad \tilde{\mathbf{\Psi}}_2=i\,\mathbf{\Psi}_4,\qquad \tilde{\mathbf{\Psi}}_3=-i\,\mathbf{\Psi}_1,\qquad \tilde{\mathbf{\Psi}}_4=-i\,\mathbf{\Psi}_2, \tag{6.10}$$

where $(U,\ \mathbf{\Psi}_1,\ \mathbf{\Psi}_2,\ \mathbf{\Psi}_3,\ \mathbf{\Psi}_4)$ is some canonically orthonormal chiral frame of DM. Taking $\hat{\mathfrak{S}}=\hat{T}$, due to (2.36) we get $\hat{\mathfrak{T}}=\hat{\mathfrak{S}}^{-1}=\hat{T}$. Then we derive

$$\tilde{d}_{ij}=\sum_{k=1}^{4}\sum_{q=1}^{4}\hat{T}^k_i\,\hat{T}^q_j\,d_{kq}=d_{ij}, \tag{6.11}$$

$$\tilde{H}^i_j=\sum_{k=1}^{4}\sum_{q=1}^{4}\hat{T}^i_k\,\hat{T}^q_j\,H^k_q=-H^i_j, \tag{6.12}$$

$$\tilde{D}_{i\bar{j}}=\sum_{k=1}^{4}\sum_{q=1}^{4}\hat{T}^k_i\,\overline{\hat{T}^{\bar{q}}_{\bar{j}}}\,D_{k\bar{q}}=-D_{i\bar{j}}. \tag{6.13}$$

The formulas (6.11), (6.12), and (6.13) are analogous to (6.7), (6.8), and (6.9). They are derived by direct calculations. In (6.12) we see that the components of the chirality operator $\mathbf{H}$ change their signs. This means that the frame (6.10), like the frame (6.6), is an anti-chiral frame. However, unlike (6.6), it is an orthonormal frame in the sense of the definition 5.2 and it is not a self-adjoint frame in the sense of the definition 5.4. Due to (6.13) it is an *anti-self-adjoint frame*.

Definition 6.2. A frame $(\tilde{U},\ \tilde{\mathbf{\Psi}}_1,\ \tilde{\mathbf{\Psi}}_2,\ \tilde{\mathbf{\Psi}}_3,\ \tilde{\mathbf{\Psi}}_4)$ produced from some canonically orthonormal chiral frame $(U,\ \mathbf{\Psi}_1,\ \mathbf{\Psi}_2,\ \mathbf{\Psi}_3,\ \mathbf{\Psi}_4)$ by means of the formulas (6.10) is called a *T-reverse anti-chiral frame* of the Dirac bundle DM.

Theorem 6.2. *A frame* $(\tilde{U},\, \tilde{\mathbf{\Psi}}_1,\, \tilde{\mathbf{\Psi}}_2,\, \tilde{\mathbf{\Psi}}_3,\, \tilde{\mathbf{\Psi}}_4)$ *is a* T*-reverse anti-chiral frame of the Dirac bundle* DM *if and only if the components of the basic spin-tensorial fields* $\mathbf{d}$, $\mathbf{H}$, *and* $\mathbf{D}$ *in this frame are given by the formulas*

$$\tilde{d}_{ij}=d_{ij}, \qquad\qquad \tilde{H}^i_j=-H^i_j, \qquad\qquad \tilde{D}_{i\bar{j}}=-D_{i\bar{j}},$$

where d_{ij}, H^i_j, $D_{i\bar{j}}$ *are taken from the matrices* (5.2), (5.3), *and* (5.4) *respectively.*

In the last step of our geometrization scheme we take the product $\hat{Q}=\hat{P}\cdot\hat{T}$. Due to our choice of positive signs in both formulas (2.37) we get

$$\hat{Q}=\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3=\begin{Vmatrix} i & 0 & 0 & 0\\ 0 & i & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & 0 & 0 & -i\end{Vmatrix}. \tag{6.14}$$

The product (6.14) is taken from (2.24). By setting $\hat{\mathfrak{S}}=\hat{Q}$ in (4.14) we get

$$\tilde{\mathbf{\Psi}}_1=i\,\mathbf{\Psi}_1, \qquad \tilde{\mathbf{\Psi}}_2=i\,\mathbf{\Psi}_2, \qquad \tilde{\mathbf{\Psi}}_3=-i\,\mathbf{\Psi}_3, \qquad \tilde{\mathbf{\Psi}}_4=-i\,\mathbf{\Psi}_4. \tag{6.15}$$

From (2.38) we derive $\hat{Q}^2=-\mathbf{1}$. Therefore, taking $\hat{\mathfrak{S}}=\hat{Q}$, we get $\hat{\mathfrak{T}}=\hat{\mathfrak{S}}^{-1}=-\hat{Q}$. Then from (4.28) we derive the relationships analogous to (6.11), (6.12), and (6.13):

$$\tilde{d}_{ij}=\sum_{k=1}^{4}\sum_{q=1}^{4}\hat{Q}^k_i\,\hat{Q}^q_j\,d_{kq}=-d_{ij}, \tag{6.16}$$

$$\tilde{H}^i_j=-\sum_{k=1}^{4}\sum_{q=1}^{4}\hat{Q}^i_k\,\hat{Q}^q_j\,H^k_q=H^i_j, \tag{6.17}$$

$$\tilde{D}_{i\bar{j}}=\sum_{k=1}^{4}\sum_{q=1}^{4}\hat{Q}^k_i\,\overline{\hat{Q}^{\bar{q}}_{\bar{j}}}\,D_{k\bar{q}}=-D_{i\bar{j}}. \tag{6.18}$$

The formulas (6.16), (6.17), and (6.18) mean that the frame (6.15) is an anti-orthonormal, chiral, and anti-self-adjoint frame of DM.

Definition 6.3. A frame $(\tilde{U},\, \tilde{\mathbf{\Psi}}_1,\, \tilde{\mathbf{\Psi}}_2,\, \tilde{\mathbf{\Psi}}_3,\, \tilde{\mathbf{\Psi}}_4)$ produced from some canonically orthonormal chiral frame $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2,\, \mathbf{\Psi}_3,\, \mathbf{\Psi}_4)$ by means of the formulas (6.15) is called a *PT-reverse chiral frame* of the Dirac bundle DM.

Theorem 6.3. *A frame* $(\tilde{U},\, \tilde{\mathbf{\Psi}}_1,\, \tilde{\mathbf{\Psi}}_2,\, \tilde{\mathbf{\Psi}}_3,\, \tilde{\mathbf{\Psi}}_4)$ *is a* PT*-reverse chiral frame of the Dirac bundle* DM *if and only if the components of the basic spin-tensorial fields* $\mathbf{d}$, $\mathbf{H}$, *and* $\mathbf{D}$ *in this frame are given by the formulas*

$$\tilde{d}_{ij}=-d_{ij}, \qquad\qquad \tilde{H}^i_j=H^i_j, \qquad\qquad \tilde{D}_{i\bar{j}}=-D_{i\bar{j}},$$

where d_{ij}, H^i_j, $D_{i\bar{j}}$ *are taken from the matrices* (5.2), (5.3), *and* (5.4) *respectively.*

Thus, the geometrization of the group homomorphism (2.39) is complete. The following diagram illustrates the frame association for frames in DM and TM:

Canonically orthonormal chiral frames	$\rightarrow$	Positively polarized right orthonormal frames
P-reverse anti-chiral frames	$\rightarrow$	Positively polarized left orthonormal frames
T-reverse anti-chiral frames	$\rightarrow$	Negatively polarized right orthonormal frames
PT-reverse chiral frames	$\rightarrow$	Negatively polarized left orthonormal frames

(6.19)

Transition matrices relating frames in the right column of the diagram (6.19) form the group $\operatorname{O}(1,3,\mathbb{R})$. For frames in the left column their transition matrices form the 4-dimensional complex representation of the group $\operatorname{Pin}(1,3,\mathbb{R})$.

7. Spin-tensorial interpretation of the Dirac matrices.

Let's denote by γ^i_{jk} the components of the k-th Dirac matrix γ_k and consider the matrix equality (2.22). If we denote $\hat{\mathfrak{T}} = \hat{\mathfrak{S}}^{-1}$, then we write it as

$$\sum_{k=0}^{3} S^k_m\ \gamma^i_{jk} = \sum_{r=1}^{4}\sum_{s=1}^{4} \hat{\mathfrak{S}}^i_r\ \hat{\mathfrak{T}}^s_j\ \gamma^r_{sm}. \tag{7.1}$$

Using the inverse matrix $T = S^{-1}$, from (7.1) we derive the following equality:

$$\gamma^i_{jk} = \sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{m=1}^{3} \hat{\mathfrak{S}}^i_r\ \hat{\mathfrak{T}}^s_j\ T^m_k\ \gamma^r_{sm}. \tag{7.2}$$

The formula (7.2) is a special case of the general transformation formula (4.29). It means that the components of all γ-matrices taken together define a spin-tensorial field of the type $(1,1|0,0|0,1)$. We denote it $\boldsymbol{\gamma}$. Here are the numeric values of γ^i_{jk} taken from the matrices γ_0, γ_1, γ_2, and γ_3, in (2.25), (2.26), (2.27), and (2.28):

$$\begin{aligned}
&\gamma^1_{1\,0} = 0, && \gamma^1_{2\,0} = 0, && \gamma^1_{3\,0} = 1, && \gamma^1_{4\,0} = 0,\\
&\gamma^2_{1\,0} = 0, && \gamma^2_{2\,0} = 0, && \gamma^2_{3\,0} = 0, && \gamma^2_{4\,0} = 1,\\
&\gamma^3_{1\,0} = 1, && \gamma^3_{2\,0} = 0, && \gamma^3_{3\,0} = 0, && \gamma^3_{4\,0} = 0,\\
&\gamma^4_{1\,0} = 0, && \gamma^4_{2\,0} = 1, && \gamma^4_{3\,0} = 0, && \gamma^4_{4\,0} = 0,
\end{aligned} \tag{7.3}$$

$$\begin{aligned}
&\gamma^1_{1\,1} = 0, && \gamma^1_{2\,1} = 0, && \gamma^1_{3\,1} = 0, && \gamma^1_{4\,1} = 1,\\
&\gamma^2_{1\,1} = 0, && \gamma^2_{2\,1} = 0, && \gamma^2_{3\,1} = 1, && \gamma^2_{4\,1} = 0,\\
&\gamma^3_{1\,1} = 0, && \gamma^3_{2\,1} = -1, && \gamma^3_{3\,1} = 0, && \gamma^3_{4\,1} = 0,\\
&\gamma^4_{1\,1} = -1, && \gamma^4_{2\,1} = 0, && \gamma^4_{3\,1} = 0, && \gamma^4_{4\,1} = 0,
\end{aligned} \tag{7.4}$$

$$\begin{aligned} &\gamma^1_{1\,2} = 0, && \gamma^1_{2\,2} = 0, && \gamma^1_{3\,2} = 0, && \gamma^1_{4\,2} = -i,\\ &\gamma^2_{1\,2} = 0, && \gamma^2_{2\,2} = 0, && \gamma^2_{3\,2} = i, && \gamma^2_{4\,2} = 0,\\ &\gamma^3_{1\,2} = 0, && \gamma^3_{2\,2} = i, && \gamma^3_{3\,2} = 0, && \gamma^3_{4\,2} = 0,\\ &\gamma^4_{1\,2} = -i, && \gamma^4_{2\,2} = 0, && \gamma^4_{3\,2} = 0, && \gamma^4_{4\,2} = 0, \end{aligned} \tag{7.5}$$

$$\begin{aligned} &\gamma^1_{1\,3} = 0, && \gamma^1_{2\,3} = 0, && \gamma^1_{3\,3} = 1, && \gamma^1_{4\,3} = 0,\\ &\gamma^2_{1\,3} = 0, && \gamma^2_{2\,3} = 0, && \gamma^2_{3\,3} = 0, && \gamma^2_{4\,3} = -1,\\ &\gamma^3_{1\,3} = -1, && \gamma^3_{2\,3} = 0, && \gamma^3_{3\,3} = 0, && \gamma^3_{4\,3} = 0,\\ &\gamma^4_{1\,3} = 0, && \gamma^4_{2\,3} = 1, && \gamma^4_{3\,3} = 0, && \gamma^4_{4\,3} = 0. \end{aligned} \tag{7.6}$$

In contrast to (4.29), we have no tilde in (7.2). Moreover, $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{T}}$ are two mutually inverse matrices of the special form (2.19), while T is produced from $\mathfrak{T}$ by means of the group homomorphism (1.1). These features mean that γ-symbols given by (7.3), (7.4), (7.5), and (7.6) should be ascribed to canonically orthonormal chiral frames of DM and to their associated positively polarized right orthonormal frames in TM (see the first line in the diagram (6.19)).

Now let's proceed to the formulas (2.32) and (2.34). These two equalities can be easily transformed to the form similar to (7.2):

$$\gamma^i_{jk} = \sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{m=1}^{3} \hat{P}^i_r\ \hat{P}^s_j\ P^m_k\ \gamma^r_{sm}, \tag{7.7}$$

$$\gamma^i_{jk} = \sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{m=1}^{3} \hat{T}^i_r\ \hat{T}^s_j\ T^m_k\ \gamma^r_{sm}. \tag{7.8}$$

The formula (7.7) is an analog of the formulas (6.7), (6.8), and (6.9), while (7.8) is an analog of the formulas (6.11), (6.12), and (6.13). From (7.7) and (7.8) one easily derives the following formula for γ-symbols:

$$\gamma^i_{jk} = -\sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{m=1}^{3} \hat{Q}^i_r\ \hat{Q}^s_j\ Q^m_k\ \gamma^r_{sm}. \tag{7.9}$$

This formula (7.9) is an analog of the formulas (6.16), (6.17), and (6.18). The matrix $\hat{Q}$ in it is taken from the formula (6.14), while the matrix Q is produced as the product of the reflection matrices (2.2):

$$Q = P\cdot T = T\cdot P = \begin{Vmatrix} -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{Vmatrix} = -\mathbf{1}.$$

Due to the formulas (7.7), (7.8), and (7.9) the scope of the formulas (7.3), (7.4), (7.5), and (7.6) can be extended so that we have the following theorem.

Theorem 7.1. *Dirac's γ-symbols are the components of a spin-tensorial field of the type $(1,1|0,0|0,1)$ given by the formulas (7.3), (7.4), (7.5), and (7.6) in any frame pair specified in the diagram (6.19).*

8. SOME CONCLUSIONS.

The P and T operators are introduced in special relativity in order to describe the spacial and time inversion operations for wave functions of elementary particles:

$$P\colon\quad \psi(t,x,y,z)\ \to\ \psi(t,-x,-y,-z), \tag{8.1}$$

$$T\colon\quad \psi(t,x,y,z)\ \to\ \psi(-t,x,y,z) \tag{8.2}$$

(see [7] and [8] for details). However, in general relativity the coordinate transformations (8.1) and (8.2) are not permitted, provided the space-time manifold M and its metric $\mathbf{g}$ are fixed. For this reason here P and T transformations are interpreted not as actual operators, but as frame transformations only. It seems to me, that in order to treat P and T as actual operators (as actual symmetries of the Nature) one should add some transformations of M and $\mathbf{g}$ performed simultaneously with the transformations (8.1) and (8.2).

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