EFFECTIVE PROCEDURE OF POINT-CLASSIFICATION FOR THE EQUATION $y'' = P + 3\,Q\,y' + 3\,R\,{y'}^2 + S\,{y'}^3$.

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For the equations of the form $y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,{y'}^2 + S(x,y)\,{y'}^3$ the problem of equivalence in the class of point transformations is considered. Effective procedure for determining the class of point equivalence for the given equation is suggested. This procedure is based on explicit formulas for the invariants.

1. Introduction.

Let's consider an ordinary differential equation $y'' = f(x,y,y')$, where $f(x,y,y')$ is the third order polynomial in y':

$$y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,{y'}^2 + S(x,y)\,{y'}^3. \tag{1.1}$$

We apply the following point-transformation to it:

$$\begin{cases} \tilde x = \tilde x(x,y),\\ \tilde y = \tilde y(x,y).\end{cases} \tag{1.2}$$

As a result of such transformation we get another equation which has the same form:

$$\tilde y'' = \tilde P(\tilde x,\tilde y) + 3\,\tilde Q(\tilde x,\tilde y)\,\tilde y' + 3\,\tilde R(\tilde x,\tilde y)\,{\tilde y'}^2 + \tilde S(\tilde x,\tilde y)\,{\tilde y'}^3. \tag{1.3}$$

Two equations (1.1) and (1.3) which are bound with the transformation (1.2) are called *point-equivalent equations*. The problem of finding criteria for detecting point-equivalence for two given equations (1.1) and (1.3) is known as *the problem of equivalence*. This problem was studied in numerous papers (see [1–23]), some of them are classical papers and others are modern ones. Results of these papers were summed up in [24]. In that paper the complete description of point-equivalence classes for the equations of the form (1.1) is given.

Some special classifying parameters play the key role in describing point-equivalence classes. Some of them are scalar invariants for the equation (1.1), others are the components of pseudotensorial fields of various weights.

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Definition 1.1. Pseudotensorial field of the type (r,s) and weight m is an indexed array $F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ that under the point transformations (1.2) is transformed as follows:

$$F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=(\det T)^m\sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\;T^{q_1}_{j_1}\,\ldots\,T^{q_s}_{j_s}\;\tilde F^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}. \tag{1.4}$$

Here in (1.4) T and S are Jacobian matrices[1] for direct and inverse point transformations respectively:

$$S=\begin{Vmatrix} x_{1.0} & x_{0.1}\\ y_{1.0} & y_{0.1}\end{Vmatrix},\qquad\qquad T=\begin{Vmatrix} \tilde x_{1.0} & \tilde x_{0.1}\\ \tilde y_{1.0} & \tilde y_{0.1}\end{Vmatrix}. \tag{1.5}$$

First two classifying parameters are determined by the coefficients of the equation (1.1) according to the following formulas:

$$\begin{aligned}
A&=P_{0.2}-2\,Q_{1.1}+R_{2.0}+2\,P\,S_{1.0}+S\,P_{1.0}-\\
&\qquad -3\,P\,R_{0.1}-3\,R\,P_{0.1}-3\,Q\,R_{1.0}+6\,Q\,Q_{0.1},\\
B&=S_{2.0}-2\,R_{1.1}+Q_{0.2}-2\,S\,P_{0.1}-P\,S_{0.1}+\\
&\qquad +3\,S\,Q_{1.0}+3\,Q\,S_{1.0}+3\,R\,Q_{0.1}-6\,R\,R_{1.0}.
\end{aligned} \tag{1.6}$$

Parameters A and B are the components of pseudovectorial field α of the weight two: $\alpha^1=B$ and $\alpha^2=-A$. The case when both these parameters are zero is known as *the case of maximal degeneration* (see [24]):

$$A=0,\qquad\qquad B=0 \tag{1.7}$$

This case is well known. Each equation for which the conditions (1.7) hold is point-equivalent to the trivial one $y''=0$. Each such equation has eight-parametric group of point symmetries isomorphic to $SL(3,\mathbb{R})$.

The conditions (1.7) that determine the case of maximal degeneration are absolutely effective. In order to check these conditions one should only differentiate coefficients of the equation (1.1) and substitute them and their derivatives into (1.6). But, apart from the case of maximal degeneration, the complete scheme of point-classification from [24] includes eight more cases: *the case of general position* and seven cases of *intermediate degeneration*. Conditions that determine these cases are much less effective. The matter is that these conditions are formulated in special variables, where the pseudovectorial field α has unitary components:

$$\alpha^1=B=1,\qquad\qquad \alpha^2=-A=0. \tag{1.8}$$

Components of any nonzero pseudovectorial field of the weight 2 can be brought to the form (1.8) in some variables. But for to find appropriate variables one should solve some system of ordinary differential equation. Theoretically this make no limitations

[1] By means of double indices in (1.5) and in what follows we denote partial derivatives. Thus for the function $f(x,y)$ by $f_{p.q}$ we denote the differentiation p-times with respect to x and q-times with respect to y.

and the procedure of point-classification was completed in [24]. In practice this step produces the great deal of inefficiency, since explicitly solvable system of ordinary differential equations is very rare event. Main goal of this paper is to eliminate this inefficient step of bringing α into the form (1.8). Then the point-classification procedure from paper [24] will become absolutely effective.

It is worth to note that analogous inefficiency took place in the theory of hydrodynamical equations which are integrable by generalized hodograph method. This inefficiency was eliminated in [25] when the integrability condition was written in an invariant tensorial form.

2. Pseudoscalar field F and pseudovectorial field β.

According to the point-classification scheme from [24], in all cases different from the case of maximal degeneration the study of the equation (1.1) starts from the construction of pseudoscalar field F of the weight 1. In special coordinates, where the conditions (1.8) hold, this field is given by the formula

$$F^5 = -P. \tag{2.1}$$

The fact that the formula (2.1) gives the pseudoscalar field of the weight 1 was checked in [24]. Now we are only to recalculate this field in arbitrary coordinates.

Theorem 2.1. *Pseudoscalar field F of the weight 1 in arbitrary coordinates is given by the formula*

$$\begin{aligned} F^5 = A\,B\,A_{0.1} + B\,A\,B_{1.0} - A^2\,B_{0.1} - B^2\,A_{1.0} -\\ - P\,B^3 + 3\,Q\,A\,B^2 - 3\,R\,A^2\,B + S\,A^3, \end{aligned} \tag{2.2}$$

where parameters A and B should be calculated by the formulas (1.6).

It is easy to check that when we substitute (1.8) into (2.2), this formula reduces to (2.1). Therefore in order to prove the theorem 2.1 we have only to prove that the formula (2.2) determines the pseudoscalar field of the weight 1. We shall not do it here, since this was done in [26]. The formula (2.2) itself was also derived in [26].

Together with F, in [24] and [26] the pseudovectorial field β of the weight 4 was defined. In special coordinates it has the following components:

$$\beta^1 = G = 3\,Q, \qquad \beta^2 = H = -3\,P. \tag{2.3}$$

In arbitrary coordinates formulas (2.3) are recalculated into the following ones:

$$\begin{aligned} G &= -B\,B_{1.0} - 3\,A\,B_{0.1} + 4\,B\,A_{0.1} + 3\,S\,A^2 - 6\,R\,B\,A + 3\,Q\,B^2,\\ H &= -A\,A_{0.1} - 3\,B\,A_{1.0} + 4\,A\,B_{1.0} - 3\,P\,B^2 + 6\,Q\,A\,B - 3\,R\,A^2. \end{aligned} \tag{2.4}$$

For raising and lowering indices we shall use the skew-symmetric matrix

$$d_{ij} = d^{ij} = \left\| \begin{matrix} 0 & 1\\ -1 & 0 \end{matrix} \right\|. \tag{2.5}$$

Components of the matrix d_{ij} form twice-covariant pseudotensorial field of the weight -1, the same values denoted by d^{ij} form twice-contravariant field of the weight 1. Pseudotensorial fields (2.5) reveal the relationship between α, β and F:

$$3\,F^5=\sum_{i=1}^{2}\alpha_i\,\beta^i=\sum_{i=1}^{2}\sum_{j=1}^{2}d_{ij}\,\alpha^i\,\beta^j. \tag{2.6}$$

All above statements concerning the fields α, β and F, as well as the explicit formulas for them, are derived on the base of transformation rules for the coefficients of the equation (1.1). In order to write these rules in a brief form let's construct the following three-index array:

$$\begin{aligned} &\theta_{111}=P, && \theta_{112}=\theta_{121}=\theta_{211}=Q,\\ &\theta_{122}=\theta_{212}=\theta_{221}=R, && \theta_{222}=S. \end{aligned} \tag{2.7}$$

Then let's raise one of the indices by means of the matrix d^{ij}:

$$\theta^k_{ij}=\sum_{r=1}^{2}d^{kr}\,\theta_{rij}. \tag{2.8}$$

Under the transformation (1.2) the components of the array θ^k_{ij} are transformed as

$$\theta^k_{ij}=\sum_{m=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2}S^k_m\,T^p_i\,T^q_j\,\tilde\theta^m_{pq}+\sum_{m=1}^{2}S^k_m\,\frac{\partial T^m_i}{\partial x^j}-\frac{\tilde\sigma_i\,\delta^k_j+\tilde\sigma_j\,\delta^k_i}{3}. \tag{2.9}$$

Here $x^1=x$, $x^2=y$, $\tilde x^1=\tilde x$, $\tilde x^2=\tilde y$, and, moreover, the following notations are made:

$$\tilde\sigma_i=\frac{\partial\ln\det T}{\partial x^i}, \qquad\qquad \delta^k_i=\begin{cases}1 & \text{for } i=k,\\ 0 & \text{for } i\neq k.\end{cases}$$

Transformation rules (2.9) for the components of the array (2.8) are quite similar to that for the components of an affine connection (see [27]). The difference consists only in the fraction with the number 3 in denominator.

3. The case of general position.

According to the scheme of point-classification from [24], the case of general position is determined by the condition $F\neq 0$. This condition is equivalent to the non-collinearity of pseudovectorial fields α and β. This fact is easily derived from the formula (2.6). Moreover, the condition $F\neq 0$ makes possible to define the following logarithmic derivatives:

$$\varphi_i=-\frac{\partial\ln F}{\partial x^i}. \tag{3.1}$$

Under the point transformations (1.2) the quantities (3.1) are transformed as

$$\varphi_i=\sum_{j=1}^{2}T^j_i\,\tilde\varphi_j-\tilde\sigma_i. \tag{3.2}$$

Relying on (3.2), we can use the quantities φ_i and the quantities (2.8) in order to construct the components of an affine connection:

$$\Gamma^k_{ij} = \theta^k_{ij} - \frac{\varphi_i\,\delta^k_j + \varphi_j\,\delta^k_i}{3}. \tag{3.3}$$

In addition to (3.3), we can define the pair of vectorial fields

$$\mathbf{X} = F^{-2}\,\alpha, \qquad\qquad \mathbf{Y} = F^{-4}\,\beta. \tag{3.4}$$

The same condition $F \neq 0$ warrants the non-collinearity of the fields $\mathbf{X}$ and $\mathbf{Y}$ from (3.4). Thus we have the moving frame in the plane of variables x and y. It is formed by vectors $\mathbf{X}$ and $\mathbf{Y}$ at each point. Connection components (3.3) define the covariant differentiation of vector fields. Now let's calculate the components of the connection (3.3) referred to the frame of the fields $\mathbf{X}$ and $\mathbf{Y}$. They are defined as the coefficients in the following expansions:

$$\begin{aligned} &\nabla_{\mathbf{X}}\mathbf{X} = \Gamma^1_{11}\,\mathbf{X} + \Gamma^2_{11}\,\mathbf{Y}, &&\nabla_{\mathbf{X}}\mathbf{Y} = \Gamma^1_{12}\,\mathbf{X} + \Gamma^2_{12}\,\mathbf{Y},\\ &\nabla_{\mathbf{Y}}\mathbf{X} = \Gamma^1_{21}\,\mathbf{X} + \Gamma^2_{21}\,\mathbf{Y}, &&\nabla_{\mathbf{Y}}\mathbf{Y} = \Gamma^1_{22}\,\mathbf{X} + \Gamma^2_{22}\,\mathbf{Y}. \end{aligned} \tag{3.5}$$

The quantities Γ^k_{ij} in the expansions (3.5) are the scalar invariants of the equation (1.1). In paper [24] they were denoted as follows:

$$\begin{aligned} &\Gamma^1_{11} = I_1, &&\Gamma^2_{11} = I_2, &&\Gamma^1_{12} = I_3, &&\Gamma^2_{12} = I_4,\\ &\Gamma^1_{21} = I_5, &&\Gamma^2_{21} = I_6, &&\Gamma^1_{22} = I_7, &&\Gamma^2_{22} = I_8. \end{aligned}$$

By differentiating these invariants along vector field $\mathbf{X}$ we get eight more invariants

$$\begin{aligned} &\mathbf{X}I_1 = I_9, &&\mathbf{X}I_2 = I_{10}, &&\mathbf{X}I_3 = I_{11}, &&\mathbf{X}I_4 = I_{12},\\ &\mathbf{X}I_5 = I_{13}, &&\mathbf{X}I_6 = I_{14}, &&\mathbf{X}I_7 = I_{15}, &&\mathbf{X}I_8 = I_{16}. \end{aligned}$$

The differentiation of the eight initial invariants along another vector field increases the number of invariants up to 24:

$$\begin{aligned} &\mathbf{Y}I_1 = I_{17}, &&\mathbf{Y}I_2 = I_{18}, &&\mathbf{Y}I_3 = I_{19}, &&\mathbf{Y}I_4 = I_{20},\\ &\mathbf{Y}I_5 = I_{21}, &&\mathbf{Y}I_6 = I_{22}, &&\mathbf{Y}I_7 = I_{23}, &&\mathbf{Y}I_8 = I_{24}. \end{aligned}$$

Repeating this procedure of differentiation along $\mathbf{X}$ and $\mathbf{Y}$, we can construct indefinite sequence of invariants by adding 16 ones in each step. According to the results of [24] and [26], the properties of this sequence of invariants divide the case of general position into three subcases:

(1) in the infinite sequence of invariants $I_k(x, y)$ one can find two functionally independent ones;
(2) invariants $I_k(x, y)$ are functionally dependent, but not all of them are constants;
(3) all invariants in the sequence $I_k(x, y)$ are constants.

In the first case the group of point symmetries of the equation (1.1) is trivial, in the second case it is one-dimensional, and in the third case it is two-dimensional (see theorems 5.1, 5.2, and 5.3 in [24]).

In order to tell which of these three cases takes place for the given particular equation of the form (1.1) we should calculate invariants I_k explicitly. Note that we can do this effectively, since the components of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ and the components of connection (3.3) are now calculated in arbitrary coordinates. We shall give the explicit formulas for eight first invariants in the sequence. These formulas were obtained in [26]:

$$
\begin{aligned}
I_3 = {} & \frac{B\,(H\,G_{1.0} - G\,H_{1.0})}{3\,F^9} - \frac{A\,(H\,G_{0.1} - G\,H_{0.1})}{3\,F^9} + \frac{H\,F_{0.1} + G\,F_{1.0}}{3\,F^5} + \\
& + \frac{B\,G^2\,P}{3\,F^9} - \frac{(A\,G^2 - 2\,H\,B\,G)\,Q}{3\,F^9} + \frac{(B\,H^2 - 2\,H\,A\,G)\,R}{3\,F^9} - \frac{A\,H^2\,S}{3\,F^9},
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
I_6 = {} & \frac{A\,(G\,A_{0.1} + H\,B_{0.1})}{12\,F^7} - \frac{B\,(G\,A_{1.0} + H\,B_{1.0})}{12\,F^7} - \frac{4\,(A\,F_{0.1} - B\,F_{1.0})}{12\,F^3} - \\
& - \frac{G\,B^2\,P}{12\,F^7} - \frac{(H\,B^2 - 2\,G\,B\,A)\,Q}{12\,F^7} - \frac{(G\,A^2 - 2\,H\,B\,A)\,R}{12\,F^7} + \frac{H\,A^2\,S}{12\,F^7}.
\end{aligned} \tag{3.7}
$$

Seventh invariant I_7 is given by the formula

$$
\begin{aligned}
I_7 = {} & \frac{G\,H\,G_{1.0} - G^2\,H_{1.0} + H^2\,G_{0.1} - H\,G\,H_{0.1}}{3\,F^{11}} + \\
& + \frac{G^3\,P + 3\,G^2\,H\,Q + 3\,G\,H^2\,R + H^3\,S}{3\,F^{11}}.
\end{aligned} \tag{3.8}
$$

Formula for the eighth invariant I_8 is similar to (3.6) and (3.7):

$$
\begin{aligned}
I_8 = {} & \frac{G\,(A\,G_{1.0} + B\,H_{1.0})}{3\,F^9} + \frac{H\,(A\,G_{0.1} + B\,H_{0.1})}{3\,F^9} - \frac{10\,(H\,F_{0.1} + G\,F_{1.0})}{3\,F^5} - \\
& - \frac{B\,G^2\,P}{3\,F^9} + \frac{(A\,G^2 - 2\,H\,B\,G)\,Q}{3\,F^9} - \frac{(B\,H^2 - 2\,H\,A\,G)\,R}{3\,F^9} + \frac{A\,H^2\,S}{3\,F^9}.
\end{aligned} \tag{3.9}
$$

Invariants I_1, I_2, I_4, and I_5 do not require separate calculations. They are expressed through that ones, which are already calculated:

$$
I_1 = -4\,I_6, \qquad I_2 = \frac{1}{3}, \qquad I_4 = 4\,I_6, \qquad I_5 = -I_8. \tag{3.10}
$$

Note that the formula (3.7) for I_6 can be simplified substantially:

$$
I_6 = \frac{A_{0.1} - B_{1.0}}{3\,F^2} - \frac{A\,F_{0.1} - B\,F_{1.0}}{3\,F^3}. \tag{3.11}
$$

Due to (3.10) the formula (3.11) simplifies the calculation for two more invariants I_1 and I_4. As for the simplification of the formulas (3.6), (3.8), and (3.9) we shall not undertake special efforts for this now.

4. Cases of intermediate degeneration.

Apart from the case of general position and the case of maximal degeneration, the scheme of point-classification from [24] includes seven cases of *intermediate degeneration* when parameters (1.6) does not vanish simultaneously. All them are characterized by the condition $F = 0$. Due to the formula (2.2) this condition is written as follows:

$$\begin{aligned} &A\,B\,A_{0.1} + B\,A\,B_{1.0} - A^2\,B_{0.1} - B^2\,A_{1.0} -\\ &\qquad - P\,B^3 + 3\,Q\,A\,B^2 - 3\,R\,A^2\,B + S\,A^3 = 0. \end{aligned} \tag{4.1}$$

The condition (4.1) is equivalent to the collinearity of pseudovectorial fields α and β. The field α is nonzero (since otherwise we would have the case of maximal degeneration). Therefore the condition of collinearity $\alpha \parallel \beta$ can be written as

$$\beta = 3\,N\,\alpha. \tag{4.2}$$

The field β has the weight 4, while the field α is of the weight 2. Therefore the coefficient of proportionality N in (4.2) is the pseudoscalar field of the weight 2. Pseudoscalar field N defined by (4.1) can be evaluated by any one of the following two formulas:

$$N = \frac{G}{3\,B}, \qquad\qquad N = -\frac{H}{3\,A}, \tag{4.3}$$

Here A, B, G and H are calculated by (1.6) and (2.4). In case of vanishing either A or B one of the formulas (4.3) gives uncertainty $0/0$, but the other remains true for to calculate N.

In special coordinates, where the conditions (1.8) hold, the field N is calculated by the first formula (4.3). Here we have

$$P = 0, \qquad\qquad N = Q. \tag{4.4}$$

The relationships (4.4) are in complete agreement with the results of [24]. In that paper, besides $N = Q$, the following two pseudoscalar fields were defined:

$$\Omega = R_{1.0} - 2\,Q_{0.1}, \qquad\qquad M = Q_{1.0} - \frac{12}{5}\,Q^2. \tag{4.5}$$

The field Ω is of the weight 1, and M is of the weight 4. Pseudovectorial field γ has the weight 3:

$$\gamma^1 = -2\,R_{1.0} + 3\,Q_{0.1} + \frac{6}{5}\,Q\,R, \qquad\qquad \gamma^2 = M. \tag{4.6}$$

Due to $F = 0$ the quantities (3.1) in all cases of intermediate degeneration are not defined. Instead of them in [24] other two quantities φ_1 and φ_2 were introduced:

$$\varphi_1 = -\frac{6}{5}\,Q, \qquad\qquad \varphi_2 = -\frac{3}{5}\,R. \tag{4.7}$$

They obey the same transformation rule (3.2) as the quantities (3.1). Formulas (4.5), (4.6) and (4.7), taken from [24], are written in special coordinates. In order to make effective we should recalculate them in arbitrary coordinates. Now we can't use the results of [26], since the cases of intermediate degeneration are not considered there.

Theorem 4.1. *If $B \neq 0$, then in any case of intermediate degeneration the parameters φ_i are defined by the formulas*

$$
\begin{aligned}
&\varphi_1 = -3\,A\,\frac{A\,S - B_{0.1}}{5\,B^2} - 3\,\frac{A_{0.1} + B_{1.0} - 3\,A\,R}{5\,B} - \frac{6}{5}\,Q,\\
&\varphi_2 = 3\,\frac{A\,S - B_{0.1}}{5\,B} - \frac{3}{5}\,R,
\end{aligned}
\tag{4.8}
$$

which hold for arbitrary curvilinear coordinates x and y on the plane.

It is easy to note that in special coordinates, where the conditions (1.8) hold, the relationships (4.8) are reduced to the form (4.7).

Proof. Let $\tilde{x}$ and $\tilde{y}$ be special coordinates, for which the conditions (1.8) are fulfilled. This can be expressed as follows:

$$
\tilde{\alpha}^1 = \tilde{B} = 1, \qquad\qquad \tilde{\alpha}^2 = -\tilde{A} = 0.
$$

Then for to calculate the parameters A and B in arbitrary (nonspecial) coordinates x and y one can use the pseudovectorial rule of transformation for the components of the field α. Using definition 1.1, we get

$$
\begin{aligned}
B = \alpha^1 &= (\det T)^2(S^1_1\,\tilde{\alpha}^1 + S^1_2\,\tilde{\alpha}^2) = \tilde{y}^2_{0.1}\,\tilde{x}_{1.0} - \tilde{y}_{1.0}\,\tilde{y}_{0.1}\,\tilde{x}_{0.1},\\
-A = \alpha^2 &= (\det T)^2(S^2_1\,\tilde{\alpha}^1 + S^2_2\,\tilde{\alpha}^2) = \tilde{y}^2_{1.0}\,\tilde{x}_{0.1} - \tilde{y}_{1.0}\,\tilde{y}_{0.1}\,\tilde{x}_{1.0}.
\end{aligned}
\tag{4.9}
$$

From $B \neq 0$ and from (4.9) we conclude that $\tilde{y}_{0.1} \neq 0$. Therefore we can resolve the equations (4.9) with respect to the derivatives $\tilde{x}_{1.0}$ and $\tilde{y}_{1.0}$:

$$
\tilde{x}_{1.0} = \frac{A}{B}\,\tilde{y}_{0.1}, \qquad\qquad \tilde{y}_{1.0} = \frac{B}{\tilde{y}^2_{0.1}} + \frac{A}{B}\,\tilde{x}_{0.1}.
\tag{4.10}
$$

By differentiating (4.10) with respect to x and y we get formulas that express $\tilde{x}_{2.0}$, $\tilde{x}_{1.1}$, $\tilde{y}_{2.0}$, and $\tilde{y}_{1.1}$ through $\tilde{x}_{0.1}$ and $\tilde{y}_{0.1}$. Substituting the obtained expressions for second order derivatives into the transformation rules for the coefficients of the equation (1.1), we derive the following relationship for S:

$$
S = \frac{\tilde{y}^2_{0.1}\,\tilde{x}_{0.2}}{B} - \frac{\tilde{y}_{0.1}\,\tilde{y}_{0.2}\,\tilde{x}_{0,1}}{B} + 3\,\frac{\tilde{y}^2_{0.1}\,\tilde{Q}\,\tilde{x}^2_{0.1}}{B} + \frac{\tilde{y}^4_{0.1}\,\tilde{S}}{B} + 3\,\frac{\tilde{y}^3_{0.1}\,\tilde{R}\,\tilde{x}_{0.1}}{B}.
\tag{4.11}
$$

Now let's use the relationship (4.11) for to express the derivative $\tilde{x}_{0.2}$ through $\tilde{x}_{0.1}$ and $\tilde{y}_{0.1}$. Upon doing this, for the coefficient R we obtain

$$
R = -\frac{5}{3}\,\frac{\tilde{y}_{0.2}}{\tilde{y}_{0.1}} + \frac{2}{3}\,\frac{B_{0.1}}{B} + \frac{A\,S}{B} + \tilde{y}_{0.1}\,\tilde{R} + 2\,\tilde{Q}\,\tilde{x}_{0.1}.
\tag{4.12}
$$

The relationship (4.12) can be used to express $\tilde{y}_{0.2}$ through $\tilde{x}_{0.1}$ and $\tilde{y}_{0.1}$. Then formula for Q can be brought to the form

$$
Q = -\frac{4}{3}\,\frac{A_{0.1}}{B} + \frac{1}{3}\,\frac{B_{1.0}}{B} + \frac{B\,\tilde{Q}}{\tilde{y}^2_{0.1}} + \frac{A\,B_{0.1}}{B^2} - \frac{A^2\,S}{B^2} + 2\,\frac{A\,R}{B}.
$$

This formula is used to express $\tilde{Q}$ through the derivative $\tilde{y}_{0.1}$. Then the formula for P can be written as

$$P=-\frac{A_{1.0}}{B}+\frac{A^3\,S}{B^3}+\frac{A\,A_{0.1}}{B^2}+\frac{A\,B_{1,0}}{B^2}-\frac{A^2\,B_{0.1}}{B^3}-3\,\frac{A^2\,R}{B^2}+3\,\frac{A\,Q}{B}.$$

This formula for P add nothing new, since it can be derived directly from the condition (4.1) when $B\neq 0$. However, the expressions for the second order derivatives $\tilde{x}_{2.0}$, $\tilde{x}_{1.1}$, $\tilde{x}_{0.2}$, $\tilde{y}_{2.0}$, $\tilde{y}_{1.1}$, and $\tilde{y}_{0.2}$ are enough to derive the formulas for φ_1 and φ_2.

For the further calculations we shall use the values of φ_1 and φ_2 in special coordinates. Here we have

$$\tilde{\varphi}_1=-\frac{6}{5}\,\tilde{Q}, \qquad\qquad \tilde{\varphi}_2=-\frac{3}{5}\,\tilde{R}.$$

Now let's substitute these values into the transformation rules (3.2) for them. We can write (3.2) as follows:

$$\varphi_i=\sum_{j=1}^{2}T^j_i\,\tilde{\varphi}_j-\frac{\partial\ln\det T}{\partial x^i}. \tag{4.13}$$

Here T is the transition matrix defined in (1.5). Substituting (4.10) and all other analogous formulas for all other second order derivatives $\tilde{x}_{2.0}$, $\tilde{x}_{1.1}$, $\tilde{x}_{0.2}$, $\tilde{y}_{2.0}$, $\tilde{y}_{1.1}$, and $\tilde{y}_{0.2}$ into (4.13), we get the formulas (4.8). All occurrences of the derivatives $\tilde{x}_{0.1}$ and $\tilde{y}_{0.1}$, and all occurrences of $\tilde{S}$, $\tilde{R}$, and $\tilde{P}$ do cancel each other during this substitution. Theorem 4.1 is proved. □

When $B=0$, formulas (4.8) do not hold. For this case we have another theorem which can be proved in a similar way.

Theorem 4.2. *If $A\neq 0$, then in any case of intermediate degeneration the parameters φ_i are defined by the formulas*

$$\begin{aligned}\varphi_1&=-3\,\frac{B\,P+A_{1.0}}{5\,A}+\frac{3}{5}\,Q,\\ \varphi_2&=3\,B\,\frac{B\,P+A_{1.0}}{5\,A^2}-3\,\frac{B_{1.0}+A_{0.1}+3\,B\,Q}{5\,A}+\frac{6}{5}\,R,\end{aligned} \tag{4.14}$$

which hold for arbitrary curvilinear coordinates x and y on the plane.

For the pseudoscalar field Ω we need not prove theorems like theorem 4.1 or theorem 4.2. In paper [24] one can find invariant definition of this field. According to [24], first, we define tensorial field ω with the following components:

$$\omega_{ij}=\frac{\partial\varphi_i}{\partial x^j}-\frac{\partial\varphi_j}{\partial x^i}. \tag{4.15}$$

Then we contract the field (4.15) with unit skew-symmetric field d from (2.5):

$$\Omega=\frac{5}{6}\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\,d^{ij}=\frac{5}{3}\left(\frac{\partial\varphi_1}{\partial y}-\frac{\partial\varphi_2}{\partial x}\right). \tag{4.16}$$

Substituting (4.8) into (4.16) we get an explicit formula for Ω in the case when $B \neq 0$:

$$\begin{aligned}\Omega=&\frac{2\,A\,B_{0.1}(A\,S-B_{0.1})}{B^3}+\frac{(2\,A_{0.1}-3\,A\,R)\,B_{0.1}}{B^2}+\\&+\frac{(B_{1.0}-2\,A_{0.1})\,A\,S}{B^2}+\frac{A\,B_{0.2}-A^2\,S_{0.1}}{B^2}-\frac{A_{0.2}}{B}+\\&+\frac{3\,A_{0.1}\,R+3\,A\,R_{0.1}-A_{1.0}\,S-A\,S_{1.0}}{B}+R_{1.0}-2\,Q_{0.1}.\end{aligned}\tag{4.17}$$

Substituting (4.8) into (4.16), we get another formula for Ω, which differs from (4.17) by the following mirror transformation:

$$\begin{aligned}&x\to y, &&y\to x,\\&P\to -S, &&S\to -P,\\&Q\to -R, &&R\to -Q.\end{aligned}\tag{4.18}$$

Transformation (4.18) can be extended to the quantities A, B, G, H, F, and N too:

$$\begin{aligned}&A\to -B, &&B\to -A,\\&G\to H, &&H\to G,\\&F\to -F, &&N\to N.\end{aligned}\tag{4.19}$$

This can be easily seen from (2.2), (2.4), and (4.3). Applying mirror transformations (4.18) and (4.19) to φ_1, φ_2, and Ω, we get

$$\varphi_1\to\varphi_2,\qquad\qquad \varphi_2\to\varphi_1,\qquad\qquad \Omega\to-\Omega.\tag{4.20}$$

On the base of the last transformation in (4.20) we can write one more formula for Ω, which holds for $A \neq 0$:

$$\begin{aligned}\Omega=&\frac{2\,B\,A_{1.0}(B\,P+A_{1.0})}{A^3}-\frac{(2\,B_{1.0}+3\,B\,Q)\,A_{1.0}}{A^2}+\\&+\frac{(A_{0.1}-2\,B_{1.0})\,B\,P}{A^2}-\frac{B\,A_{2.0}+B^2\,P_{1.0}}{A^2}+\frac{B_{2.0}}{A}+\\&+\frac{3\,B_{1.0}\,Q+3\,B\,Q_{1.0}-B_{0.1}\,P-B\,P_{0.1}}{A}+Q_{0.1}-2\,R_{1.0}.\end{aligned}\tag{4.21}$$

Formula (4.21) is mirror symmetric with respect to the formula (4.17) in the sense of the above mirror transformations.

Formulas (4.19) and (4.21) implement the effectivization of the formula (4.5) for Ω. As for the formulas (4.5) and (4.6), their effectivization require more efforts. Let's start with the affine connection given by

$$\Gamma^k_{ij}=\theta^k_{ij}-\frac{\varphi_i\,\delta^k_j+\varphi_j\,\delta^k_i}{3}.\tag{4.22}$$

Here the array θ^k_{ij} is defined by the coefficients of the equation (1.1) according to the formulas (2.7) and (2.8), while φ_i are derived either by (4.8) or (4.14). We shall use connection components (4.22) and the quantities φ_i for covariant differentiation of pseudotensorial fields:

$$\nabla_k F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^k}+\sum_{n=1}^{r}\sum_{v_n=1}^{2}\Gamma^{i_n}_{k\,v_n}\,F^{i_1\ldots\,v_n\ldots\,i_r}_{j_1\ldots\,j_s}-\\ -\sum_{n=1}^{s}\sum_{w_n=1}^{2}\Gamma^{w_n}_{k\,j_n}\,F^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s}+m\,\varphi_k\,F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}. \tag{4.23}$$

This formula (4.23) is the definition of the covariant derivative ∇_k for pseudotensorial field F of the type (r,s) and weight m (for more details see [28]). As a result of applying (4.23) we get pseudotensorial field ∇F of the type $(r,s+1)$ and weight m.

Let's apply the operation of covariant differentiation to the pseudoscalar field N of the weight 2. This gives us pseudocovectorial field ∇N of the weight 2. Here are the components of the field ∇N:

$$\nabla_1 N=N_{1.0}+2\,\varphi_1\,N,\qquad\qquad \nabla_2 N=N_{0.1}+2\,\varphi_2\,N. \tag{4.24}$$

Let's denote by ξ the pseudovectorial field obtained from ∇N by raising index by virtue of the matrix d^{ij} from (2.5):

$$\xi^i=\sum_{j=1}^{2}d^{ij}\,\nabla_j N. \tag{4.25}$$

This field ξ has the weight 3, which coincides with the weight of the field γ. By direct calculations from (4.24) and (4.25) in special coordinates we get

$$\xi^1=Q_{0.1}-\frac{6}{5}\,R\,Q,\qquad\qquad \xi^2=-Q_{1.0}+\frac{12}{5}\,Q^2=-M. \tag{4.26}$$

Comparing (4.26) with (4.5), (4.6), and (1.8), we derive the following relationships between fields α, γ, ξ, M, and Ω:

$$\gamma=-2\,\Omega\,\alpha-\xi,\qquad\qquad M=-\sum_{i=1}^{2}\alpha_i\,\xi^i. \tag{4.27}$$

These relationships are written in terms of natural operations of sum, tensor-product, and contraction for pseudotensorial fields. Therefore, once they are established in special coordinates, they remain true in arbitrary coordinates too. For the field M from (4.27) we obtain

$$M=-\frac{12\,A\,N\,(A\,S-B_{0.1})}{5\,B}-A\,N_{0.1}+\frac{24}{5}\,A\,N\,R-\\ -\frac{6}{5}\,N\,A_{0.1}-\frac{6}{5}\,N\,B_{1.0}+B\,N_{1.0}-\frac{12}{5}\,B\,N\,Q, \tag{4.28}$$

$$M=-\frac{12\,B\,N\,(B\,P+A_{1.0})}{5\,A}+B\,N_{1.0}+\frac{24}{5}\,B\,N\,Q+\\ +\frac{6}{5}\,N\,B_{1.0}+\frac{6}{5}\,N\,A_{0.1}-A\,N_{0.1}-\frac{12}{5}\,A\,N\,R. \tag{4.29}$$

Formula (4.29) is derived from (4.28) by means of mirror transformations (4.18) and (4.19). It holds for $A \neq 0$, while the initial formula (4.28) holds for $B \neq 0$.

For the components of pseudovectorial field γ from (4.27) we derive the following relationships, which give the required effectivization for (4.6) when $B \neq 0$:

$$\gamma^1 = -\frac{6\,N\,(A\,S - B_{0.1})}{5\,B} - N_{0.1} + \frac{6}{5}\,N\,R - 2\,\Omega\,B, \tag{4.30}$$

$$\begin{aligned}\gamma^2 = &-\frac{6\,A\,N\,(A\,S - B_{0.1})}{5\,B^2} + \frac{18\,N\,A\,R}{5\,B} - \\ &-\frac{6\,N\,(A_{0.1} + B_{1.0})}{5\,B} + N_{1.0} - \frac{12}{5}\,N\,Q + 2\,\Omega\,A.\end{aligned} \tag{4.31}$$

Mirror symmetric formulas for γ, which hold for $A \neq 0$, have the form

$$\begin{aligned}\gamma^1 = &-\frac{6\,B\,N\,(B\,P + A_{1.0})}{5\,A^2} + \frac{18\,N\,B\,Q}{5\,A} + \\ &+\frac{6\,N\,(B_{1.0} + A_{0.1})}{5\,A} - N_{0.1} - \frac{12}{5}\,N\,R - 2\,\Omega\,B,\end{aligned} \tag{4.32}$$

$$\gamma^2 = -\frac{6\,N\,(B\,P + A_{1.0})}{5\,A} + N_{1.0} + \frac{6}{5}\,N\,Q + 2\,\Omega\,A. \tag{4.33}$$

The mirror transformations themselves for the components of γ are written as

$$\gamma^1 \to -\gamma^2, \qquad\qquad \gamma^2 \to -\gamma^1. \tag{4.34}$$

We can derive (4.34) by comparing (4.30) and (4.31) with (4.32) and (4.33) and taking into account (4.18) and (4.19).

5. First case of intermediate degeneration.

First case of intermediate degeneration is distinguished from other cases of intermediate degeneration by the condition $M \neq 0$. This condition $M \neq 0$ is equivalent to the condition of non-collinearity $\alpha \nparallel \gamma$ for the pseudovectorial fields α and γ. Moreover, from $M \neq 0$ one can derive $N \neq 0$. This can be easily seen either from (4.28) or from (4.29). Therefore we immediately get two scalar invariants

$$I_1 = \frac{M}{N^2}, \qquad\qquad I_2 = \frac{\Omega^2}{N}. \tag{5.1}$$

Now they can be calculated explicitly in arbitrary coordinates on the base of the above formulas for M, N, and Ω. In order to define third invariant I_3 in [24] the following expansions were considered:

$$\begin{aligned}&\nabla_\alpha\alpha = \Gamma^1_{11}\,\alpha + \Gamma^2_{11}\,\gamma, &&\nabla_\alpha\gamma = \Gamma^1_{12}\,\alpha + \Gamma^2_{12}\,\gamma,\\ &\nabla_\gamma\alpha = \Gamma^1_{21}\,\alpha + \Gamma^2_{21}\,\gamma, &&\nabla_\gamma\gamma = \Gamma^1_{22}\,\alpha + \Gamma^2_{22}\,\gamma.\end{aligned} \tag{5.2}$$

The use of these expansions is correct, since in the first case of intermediate degeneration the fields α and γ are non-collinear and they form moving frame on the plane. Let's denote by C and D the components of the field γ:

$$C = \gamma^1, \qquad\qquad D = \gamma^2. \tag{5.3}$$

In terms of (5.3) for the coefficient Γ^1_{22} in (5.2) one can derive

$$\begin{aligned}\Gamma^1_{22} =&\frac{C\,D\,(C_{1.0}-D_{0.1})}{M}+\frac{D^2\,C_{0.1}-C^2\,D_{1.0}}{M}+\\&+\frac{P\,C^3+3\,Q\,C^2\,D+3\,R\,C\,D^2+S\,D^3}{M}.\end{aligned}\tag{5.4}$$

The quantity Γ^1_{22} given by (5.4) is a pseudoscalar field of the weight 4. According to [24], it defines the third basic invariant for the first case of intermediate degeneration

$$I_3=\frac{\Gamma^1_{22}\,Q^2}{M}.\tag{5.5}$$

By differentiating invariants I_1, I_2, and I_3 along pseudovectorial fields α and γ we get six new invariants I_5, I_6, I_7, I_8, I_9, and I_{10}:

$$\begin{aligned}&I_4=\frac{\nabla_\alpha I_1}{N}, &&I_5=\frac{\nabla_\alpha I_2}{N}, &&I_6=\frac{\nabla_\alpha I_3}{N},\\ &I_7=\frac{(\nabla_\gamma I_1)^2}{N^3}, &&I_8=\frac{(\nabla_\gamma I_2)^2}{N^3}, &&I_9=\frac{(\nabla_\gamma I_3)^2}{N^3}.\end{aligned}$$

Repeating this procedure more and more, we can form an indefinite sequence of scalar invariants I_1, I_2, I_3, ... , adding 6 ones in each step.

The number of basic invariants in first case of intermediate degeneration is less by one than in case of general position, these are the invariants I_1 and I_2 from (5.1) and the invariant I_3 from (5.5). Other coefficients in (5.4) do not change the number of basic invariants, since values of them are trivial in most:

$$\Gamma^2_{11}=\Gamma^1_{21}=0, \qquad\qquad \Gamma^1_{11}=\Gamma^2_{21}=-\frac{3}{5}\,N.$$

For nontrivial coefficients $\Gamma^2_{22}=-\Gamma^1_{12}$ and Γ^2_{12} the following relations were derived:

$$I_1\,\Gamma^2_{12}=I_4\,N-\frac{3}{5}\,I_1\,N-2\,I_1^2\,N,\tag{5.6}$$

$$\begin{aligned}&\left(I_1\,\Gamma^2_{22}\right)^4+\left(I_7\,N^3\right)^2+\left(16\,I_2\,N^3\,{I_1}^4\right)^2=\\&=32\,I_7\,N^6\,I_2\,{I_1}^4+2\left(I_7\,N^3+16\,I_2\,N^3\,{I_1}^4\right)\left(I_1\,\Gamma^2_{22}\right)^2.\end{aligned}\tag{5.7}$$

Due to the relations (5.6) and (5.7) derived in [24] the coefficients Γ^2_{22}, Γ^1_{12}, and Γ^2_{12} can be expressed through the invariants from the above sequence and the field N.

Here, like in the case of general position, the structure of invariants in the sequence I_1, I_2, I_3, ... distinguishes three different subcases:

(1) in the infinite sequence of invariants $I_k(x,y)$ one can find two functionally independent ones;
(2) invariants $I_k(x,y)$ are functionally dependent, but not all of them are constants;
(3) all invariants in the sequence $I_k(x,y)$ are constants.

In the first case the group of point symmetries of the equation (1.1) is trivial, in the second case it is one-dimensional, and in the third case it is two-dimensional. When two-dimensional, this algebra is Abelian if and only if

$$I_1 = -\frac{12}{5}, \qquad\qquad I_2 = 0. \tag{5.8}$$

This is the result from [24]. Now it is absolutely effective, since conditions (5.8) can be tested without transforming the equation (1.1) to special coordinates.

6. Second case of intermediate degeneration.

Remember, that for all cases of intermediate degeneration the parameters A and B do not vanish simultaneously, while parameter F from (2.2) is zero identically. The above first case of intermediate degeneration was distinguished by the additional condition $M \neq 0$. For the second case of intermediate degeneration this additional condition is replaced by the following two relationships:

$$M = 0, \qquad\qquad N \neq 0. \tag{6.1}$$

Additional condition (6.1) distinguishes four cases: second, third, fourth and fifth case of intermediate degeneration. In each of these four cases one can choose special variables that realize the conditions (1.8), and for which the the coefficients P and Q in (1.1) are brought to the form

$$P = 0, \qquad\qquad Q = -\frac{5}{12\,x}. \tag{6.2}$$

In these variables the third coefficient R in (1.1) also has the special form defined by two arbitrary functions $r(y)$ and $c(y)$:

$$R = r(y) + c(y)\,|x|^{-1/4}. \tag{6.3}$$

Second case of intermediate degeneration is distinguished from third, fourth and fifth cases by additional condition

$$c(y) \neq 0 \tag{6.4}$$

for the function $c(y)$ in (6.3).

The conditions (6.1) are written in arbitrary variables, they do not need effectivization. Therefore we are only to make effective the condition (6.4). Let's calculate the pseudoscalar field Ω in special coordinates. Taking into account (6.2), (6.3), and (1.8), we use the formula (4.5):

$$\Omega = R_{1.0} - 2\,Q_{0.1} = -c(y)\,\frac{|x|^{-1/4}}{4\,x}. \tag{6.5}$$

Comparing (6.4) and (6.5), we conclude that condition (6.4) can be written as

$$\Omega \neq 0. \tag{6.6}$$

This condition (6.6) is an effective form for the condition (6.4). Being fulfilled in special coordinates, it remains true in any other coordinates too. This is due to pseudoscalar rule of transformation for Ω.

Let $F = 0$ and let the conditions (6.1) and (6.6) be fulfilled, i. e. we are in the second case of intermediate degeneration. Then at the expense of further specialization of the choice of variables we can bring the condition (6.4) to the form $c(y) = 1$. As a result of this the relationship (6.3) will have the form

$$R = r(y) + |x|^{-1/4}, \tag{6.7}$$

and for the parameter S we can get the following explicit expression:

$$S = \sigma(y)\,|x|^{5/4} - 4\,s(y)\,x + \frac{4}{3}\,|x|^2 - 12\,\frac{r(y)\,|x|^{7/4}}{x} - 4\,\frac{|x|^{3/2}}{x}. \tag{6.8}$$

(for more details see [24]). Here $s(y)$ and $\sigma(y)$ are two arbitrary functions of one variable. The above formulas (6.2), (6.7), and (6.8) define the canonical form of the equation (1.1) in the second case of intermediate degeneration. Algebra of point symmetries for such equation is described by the following theorem from [24].

Theorem 6.1. *In the second case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if parameters* $r(y)$, $s(y)$, *and* $\sigma(y)$ *are identically constant:*

$$r'(y) = 0, \qquad s'(y) = 0, \qquad \sigma'(y) = 0. \tag{6.9}$$

If at least one of the conditions (6.9) *fails, then corresponding algebra of point symmetries is trivial.*

Conditions (6.9) determining the structure of the algebra of point symmetries in the theorem 6.1 are written in special variables. Therefore they require effectivization. Remember, that the condition $M \neq 0$ is equivalent to the non-collinearity of pseudovectorial fields α and γ. Conversely, from $M = 0$ we derive $\gamma \parallel \alpha$ and remember that $\alpha \neq 0$. Then the proportionality factor relating these two field $\gamma = \Lambda\,\alpha$ defines one more pseudoscalar field of the weight 1. The field Λ can be calculated by one of the following formulas similar to (4.3):

$$\Lambda = \frac{C}{B}, \qquad \Lambda = -\frac{D}{A}. \tag{6.10}$$

Here by C and D the components of the field γ are denoted (see (5.3)). Formulas (6.10) are effective, they are applicable in arbitrary coordinates. It would be convenient to write them in more explicit form:

$$\Lambda = -\frac{6\,N\,(A\,S - B_{0.1})}{5\,B^2} - \frac{N_{0.1}}{B} + \frac{6\,N\,R}{5\,B} - 2\,\Omega, \tag{6.11}$$

$$\Lambda = \frac{6\,N\,(B\,P + A_{1.0})}{5\,A^2} - \frac{N_{1.0}}{A} - \frac{6\,N\,Q}{5\,A} - 2\,\Omega. \tag{6.12}$$

These two formulas (6.11) and (6.12) are easily derived from (4.30) and (4.33). They are mirror symmetric to each other, while mirror transformation for Λ is written as follows: $\Lambda \to -\Lambda$.

Let's calculate the fields Ω and Λ in special coordinates that were introduced above for second case of special degeneration. From (6.2), (6.7), and (6.8) we derive

$$\Omega=-\frac{|x|^{-1/4}}{4\,x},\qquad\qquad \Lambda=-\frac{r(y)}{2\,x}. \tag{6.13}$$

By means of N, Ω, and Λ let's construct one more field

$$I_1=\frac{\Lambda^{12}}{\Omega^8\,N^2}. \tag{6.14}$$

The weight of the field (6.14) appears to be zero: $12-8-2\cdot 2=0$. Thus, the field I_1 is a scalar invariant of the equation (1.1). It is not difficult to calculate this field in special coordinates:

$$I_1=\frac{2304}{25}\,r(y)^{12}. \tag{6.15}$$

By comparing (6.15) with (6.9) we conclude that first of the conditions (6.9) can be written in the following invariant form:

$$I_1=\operatorname{const}. \tag{6.16}$$

The condition (6.16) can be checked effectively without transforming the equation (1.1) to special coordinates. Effectivization of the rest two conditions in (6.9) requires some additional efforts.

7. Curvature tensor and additional fields.

Let $F=0$ and $M=0$, while parameters A and B do not vanish simultaneously. This corresponds to any case of intermediate degeneration, except for the first. Theorems 4.1 and 4.2 give us effective formulas for the parameters φ_1 and φ_2. Then these parameters are used to determine the connection components (4.22). In turn they determine the field of curvature tensor:

$$R^k_{qij}=\frac{\partial\Gamma^k_{jq}}{\partial u^i}-\frac{\partial\Gamma^k_{iq}}{\partial u^j}+\sum^2_{s=1}\Gamma^k_{is}\Gamma^s_{jq}-\sum^2_{s=1}\Gamma^k_{js}\Gamma^s_{iq}. \tag{7.1}$$

Curvature tensor (7.1) is skew symmetric with respect to the last pair of indices i and j. I two-dimensional geometry such tensor can be decomposed as $R^k_{qij}=R^k_q\,d_{ij}$. Here R^k_q is pseudotensorial field of the weight 1. It can be calculated by the formula

$$R^k_q=\frac{1}{2}\sum^2_{i=1}\sum^2_{j=1}R^k_{qij}\,d^{ij}. \tag{7.2}$$

For to study the property of the pseudotensorial field (7.2) let's calculate its components in special coordinates, where the conditions (1.8) hold. In such coordinates the field α has unitary components: $\alpha^1=1$, $\alpha^2=0$. The conditions $F=0$ and $M=0$ are written as $P=0$ and $Q_{1.0}=5/12\,Q^2$, and the parameters φ_1 and φ_2 are given by the formulas (4.7). Taking into account all these circumstances, we can calculate the components of the field (7.2) in explicit form. It is remarkable that due to $M = 0$ the matrix R^k_q appears to be upper-triangular in special coordinates:

$$R^k_q = \begin{Vmatrix} R^1_1 & R^1_2 \\ 0 & R^2_2 \end{Vmatrix}. \tag{7.3}$$

Eigenvalues $\lambda_1 = R^1_1$ and $\lambda_2 = R^2_2$ of the matrix (7.3) are the pseudoscalar fields of the weight 1. They can be calculated in explicit form:

$$\lambda_1 = -\frac{3}{5}\,\Lambda, \qquad\qquad \lambda_2 = \frac{3}{5}\,\Omega + \frac{3}{5}\,\Lambda. \tag{7.4}$$

Let subtract the identity matrix multiplied by the second eigenvalue $\lambda_2 = R^2_2$ from the matrix (7.3). As a result we have the matrix

$$P^k_q = R^k_q - \lambda_2\,\delta^k_q = \begin{Vmatrix} (\lambda_1 - \lambda_2) & R^1_2 \\ & \\ 0 & 0 \end{Vmatrix}, \tag{7.5}$$

which defines another pseudotensorial field of the weight 1. Let $\mathbf{X}$ be some arbitrary vector-field with components X^1 and X^2. Let's contract it with (7.5). Then we obtain pseudovectorial field $P\mathbf{X}$ of the weight 1, whose second component being identically zero. This means that $P\mathbf{X}$ is collinear to the field α having unitary components $\alpha^1 = 1$ and $\alpha^2 = 0$ in special coordinates:

$$P\mathbf{X} = \bigl((\lambda_1 - \lambda_2)\,X^1 + R^1_2\,X^2\bigr)\ \alpha. \tag{7.6}$$

The proportionality factor binding $P\mathbf{X}$ and α in (7.6) depends linearly on the components of the vector $\mathbf{X}$. Therefore it defines pseudocovectorial field of the weight -1 with the following components:

$$\omega_1 = \lambda_1 - \lambda_2, \qquad\qquad \omega_2 = R^1_2. \tag{7.7}$$

Components ω_1 and ω_2 from (7.7) can be calculated explicitly:

$$\omega_1 = -\frac{3}{5}\,\Omega - \frac{6}{5}\,\Lambda, \qquad\qquad \omega_2 = S_{1.0} - \frac{6}{5}\,R_{0.1} + \frac{12}{5}\,S\,Q - \frac{54}{25}\,R^2. \tag{7.8}$$

Formulas (7.4) for the eigenvalues of the field (7.3) need no effectivization. They do not change in arbitrary coordinates if we take into account the formulas (4.17), (4.21), (6.11), and (6.12) for the fields Ω and Λ. But the formulas (7.8) should be recalculated for the case of arbitrary coordinates. In order to do it note that the formulas (7.1) and (7.2) hold for arbitrary coordinates. Then formula (7.5) in non-special coordinates is written as

$$P^k_q = R^k_q - \lambda_2\,\delta^k_q = \begin{Vmatrix} R^1_1 - \lambda_2 & R^1_2 \\ & \\ R^2_1 & R^2_2 - \lambda_2 \end{Vmatrix}.$$

This is due to the fact that matrix R^k_q in arbitrary coordinates isn't upper-triangular. However, the collinearity of the fields $P\mathbf{X}$ and α does not depend on the choice of coordinates. Hence the formula (7.6) in arbitrary coordinates is written as

$$P\mathbf{X}=\begin{Vmatrix} R^1_1-\lambda_2 & R^1_2\\ \\ R^2_1 & R^2_2-\lambda_2\end{Vmatrix}\cdot\begin{Vmatrix}X^1\\ \\ X^2\end{Vmatrix}=(\omega_1\,X^1+\omega_2\,X^2)\cdot\begin{Vmatrix}B\\ \\ -A\end{Vmatrix}. \tag{7.9}$$

From (7.9) we easily extract the required effective formulas for ω_1 and ω_2:

$$\omega_1=\frac{R^1_1-\lambda_2}{B}, \qquad\qquad \omega_2=\frac{R^1_2}{B}. \tag{7.10}$$

Formulas (7.10) hold for $B\neq 0$. For the case $A\neq 0$ we can write mirror symmetric formulas. They are the following ones:

$$\omega_1=-\frac{R^2_1}{A}, \qquad\qquad \omega_2=\frac{\lambda_2-R^2_2}{A}. \tag{7.11}$$

Upon explicit calculation of the components of matrix R^k_q and upon substituting them into (7.10) for the field ω in arbitrary coordinates we get

$$\begin{aligned} \omega_1=&-\frac{6\,\Lambda+3\,\Omega}{5\,B}+\frac{5\,A\,S_{1.0}-6\,A\,R_{0.1}+12\,Q\,A\,S}{5\,B^2}-\frac{54}{25}\,\frac{A\,R^2}{B^2}+\\ &+\frac{2\,A\,A_{0.1}\,S+A\,B_{1.0}\,S+A^2\,S_{0.1}-A\,B_{0.2}}{5\,B^3}-\frac{12\,A^2\,S\,R}{25\,B^3}+\\ &+\frac{3\,A\,R\,B_{0.1}}{25\,B^3}+\frac{6\,A\,B^2_{0.1}+6\,A^3\,S^2-12\,A^2\,B_{0.1}\,S}{25\,B^4}, \end{aligned} \tag{7.12}$$

$$\begin{aligned} \omega_2=&\,\frac{12\,S\,Q}{5\,B}-\frac{54}{25}\,\frac{R^2}{B}+\frac{S_{1.0}}{B}-\frac{6\,R_{0.1}}{5\,B}+\frac{S\,B_{1.0}+A\,S_{0.1}-B_{0.2}}{5\,B^2}+\\ &+\frac{2\,A_{0.1}\,S}{5\,B^2}-\frac{3\,R\,B_{0.1}+12\,S\,A\,R}{25\,B^2}+\frac{6\,A^2\,S^2-12\,B_{0.1}\,A\,S+6\,B^2_{0.1}}{25\,B^3}. \end{aligned} \tag{7.13}$$

Formulas (7.12) and (7.13) hold for $B\neq 0$. For the case $A\neq 0$ we have formulas that are derived from (7.11):

$$\begin{aligned} \omega_1=&\,\frac{12\,P\,R}{5\,A}-\frac{54}{25}\,\frac{Q^2}{A}-\frac{P_{0.1}}{A}+\frac{6\,Q_{1.0}}{5\,A}-\frac{P\,A_{0.1}+B\,P_{1.0}+A_{2.0}}{5\,A^2}-\\ &-\frac{2\,P\,B_{1.0}}{5\,A^2}+\frac{3\,Q\,A_{1.0}-12\,P\,B\,Q}{25\,A^2}+\frac{6\,B^2\,P^2+12\,B\,P\,A_{1.0}+6\,A^2_{1.0}}{25\,A^3}, \end{aligned} \tag{7.14}$$

$$\begin{aligned} \omega_2=&\,\frac{6\,\Lambda+3\,\Omega}{5\,A}+\frac{6\,B\,Q_{1.0}+12\,R\,B\,P-5\,B\,P_{0.1}}{5\,A^2}-\frac{54}{25}\,\frac{B\,Q^2}{A^2}-\\ &-\frac{2\,B\,B_{1.0}\,P+B\,P\,A_{0.1}+B^2\,P_{1.0}+B\,A_{2.0}}{5\,A^3}-\frac{12\,B^2\,P\,Q}{25\,A^3}+\\ &+\frac{3\,B\,Q\,A_{1.0}}{25\,A^3}+\frac{6\,B\,A^2_{1.0}+6\,B^3\,P^2+12\,B^2\,P\,A_{1.0}}{25\,A^4}. \end{aligned} \tag{7.15}$$

Formulas (7.14) and (7.15) are mirror symmetric with respect to the formulas (7.12) and (7.13). Comparing these two pairs of formulas, we derive the following mirror transformations for ω_1 and ω_2:

$$\omega_1 \to -\omega_2, \qquad\qquad \omega_2 \to -\omega_1. \tag{7.16}$$

Mirror transformations (7.16) are to be considered as the expansion for the transformations (4.18), (4.19), (4.20), and (4.34).

Let $F = 0$ and $M = 0$ as before and suppose that A and B do not vanish simultaneously. Under these assumptions we have two pseudoscalar fields Ω and Λ of the weight 1. In special coordinates they are defined by

$$\Omega = R_{1.0} - 2\,Q_{0.1}, \qquad\qquad \Lambda = -2\,R_{1.0} + 3\,Q_{0.1} + \frac{6}{5}\,Q\,R. \tag{7.17}$$

The effectivization for the formulas (7.17) has been already done in form of relationships (4.17), (4.21), (6.11), and (6.12). Let's calculate the covariant differentials $\nabla\Omega$ and $\nabla\Lambda$ for Ω and Λ. These are pseudocovectorial fields of the weight 1. From (7.17) and (4.23) for the components of $\nabla\Omega$ and $\nabla\Lambda$ in special coordinates we derive

$$\nabla_1\Omega = \frac{9}{5}\,Q\,\Omega, \qquad\qquad \nabla_2\Omega = \Omega_{0.1} - \frac{3}{5}\,R\,\Omega, \tag{7.18}$$

$$\nabla_1\Lambda = \frac{6}{5}\,Q\,\Lambda, \qquad\qquad \nabla_2\Lambda = \Lambda_{0.1} - \frac{3}{5}\,R\,\Lambda. \tag{7.19}$$

Let's compare (7.18) and (7.19) with the formulas (7.8) for the components of the field ω, which also has the weight 1. Then let's construct the field

$$w = N\,\omega + \nabla\Lambda + \frac{1}{3}\,\nabla\Omega. \tag{7.20}$$

This field (7.20) has the weight 1. It's remarkable that its first component in special coordinates is zero: $w_1 = 0$. Hence w is collinear to pseudocovectorial field α with components $\alpha_1 = 0$ and $\alpha_2 = 1$. Let's denote by K the proportionality factor in $w = K\,\alpha$. Then K is a scalar field (field of the weight 0). In special coordinates it is calculated as follows:

$$\begin{aligned} K = \Lambda_{0.1} - \frac{3}{5}\,R\,\Lambda + \frac{1}{3}\,\Omega_{0.1} - \frac{1}{5}\,R\,\Omega +\\ + Q\,S_{1.0} - \frac{6}{5}\,Q\,R_{0.1} + \frac{12}{5}\,S\,Q^2 - \frac{54}{25}\,Q\,R^2. \end{aligned} \tag{7.21}$$

It's no problem to make effective (7.21), since the field (7.20) can be evaluated in arbitrary coordinates. When $B \neq 0$, we have

$$K = \frac{\Lambda_{0.1} + \Lambda\varphi_2}{B} + \frac{\Omega_{0.1} + \Omega\varphi_2}{3\,B} + \frac{N\,\omega_2}{B}. \tag{7.22}$$

Here Λ and Ω are calculated by the formulas (6.11) and (4.17), N is defined by the first relationship (4.3), parameter φ_2 is given by the formula (4.8), and ω_2 is defined by the formula (7.13). For $A \neq 0$ we have the formula mirror symmetric to (7.22):

$$K = \frac{\Lambda_{1.0} + \Lambda\varphi_1}{A} + \frac{\Omega_{1.0} + \Omega\varphi_1}{3\,A} + \frac{N\,\omega_1}{A}. \tag{7.23}$$

In (7.23) fields Λ and Ω are calculated by (6.12) and (4.21), field N is defined by the second relationship (4.3), parameter φ_1 is given by (4.14), and ω_1 is calculated by the formula (7.14).

8. Algebra of symmetries in the second case of intermediate degeneration.

In special coordinates the structure of the algebra of point symmetries of the equation (1.1) for this case is described by the conditions (6.9) in theorem 6.1. One of them had been made effective in form of the condition (6.16). In order to make effective two other conditions (6.9) we shall construct some additional scalar invariants. Let's consider pseudocovectorial field ε defined by the formula analogous to (7.20):

$$\varepsilon = N\,\omega + \nabla\Lambda. \tag{8.1}$$

Field (8.1) has the weight 1. After raising indices by means of skew-symmetric matrix from (2.5) we get the pseudovectorial field ε of the weight 2. It's easy to calculate the components of this field in special coordinates:

$$\begin{aligned}&\varepsilon^1 = Q\,S_{1.0} - \frac{6}{5}\,Q\,R_{0.1} + \frac{12}{5}\,S\,Q^2 - \frac{54}{25}\,Q\,R^2 + \Lambda_{0.1} - \frac{3}{5}\,R\,\Lambda,\\ &\varepsilon^2 = \frac{3}{5}\,Q\,\Omega.\end{aligned} \tag{8.2}$$

In the second case of intermediate degeneration we have $N \neq 0$ and $\Omega \neq 0$ (see (6.1) and (6.6)). In special coordinates this yields $Q\,\Omega \neq 0$, i. e. second component of the field (8.2) is nonzero. Therefore pseudovectorial fields α and ε of the same weight 2 are non-collinear and we are able to write the expansions

$$\begin{aligned}&\nabla_\alpha\alpha = \Gamma^1_{11}\,\alpha + \Gamma^2_{11}\,\varepsilon, &&\nabla_\alpha\varepsilon = \Gamma^1_{12}\,\alpha + \Gamma^2_{12}\,\varepsilon,\\ &\nabla_\varepsilon\alpha = \Gamma^1_{21}\,\alpha + \Gamma^2_{21}\,\varepsilon, &&\nabla_\varepsilon\varepsilon = \Gamma^1_{22}\,\alpha + \Gamma^2_{22}\,\varepsilon.\end{aligned} \tag{8.3}$$

Covariant derivatives in (8.3) are determined by the connection (4.22). All coefficients Γ^k_{ij} in these expansions are the pseudoscalar fields of the weight 2. Most of them are trivial or equal to zero:

$$\Gamma^2_{11} = \Gamma^1_{21} = 0, \qquad\qquad \Gamma^1_{11} = \Gamma^2_{21} = -\frac{3}{5}\,N.$$

Others are less trivial, but, nevertheless, they can be expressed through the pseudoscalar fields K, N, Ω, and Λ:

$$\begin{aligned}&\Gamma^1_{12} = -\frac{6}{5}\,K\,N + N - \frac{6}{5}\,\Lambda^2 - 3\,\Omega\,\Lambda,\\ &\Gamma^2_{22} = \frac{9}{5}\,K\,N - \frac{6}{5}\,\Omega^2 - \frac{3}{5}\,\Omega\,\Lambda.\end{aligned}$$

The only new field in (8.3) is the field Γ^1_{22}. Explicit formula for Γ^1_{22} in special coordinates contains 56 summands. We shall not write it here. Instead of this, we shall describe the effective algorithm to calculate Γ^1_{22} in arbitrary coordinates.

Let's start with the relationships (8.2) for the components of pseudovectorial field ε. Their effectivization is based on the formula (8.1) for this field:

$$\varepsilon^1 = N\,\omega_2 + \Omega_{0.1} + \varphi_2\,\Omega, \qquad\qquad \varepsilon^2 = -N\,\omega_1 - \Omega_{1.0} + \varphi_1\,\Omega. \tag{8.4}$$

Here ω_1 and ω_2 are calculated by the formulas (7.12), (7.13), (7.14), and (7.15), parameters φ_1 and φ_2 are given by (4.8) and (4.14). Denote by C and D the components of the field (8.4): $\varepsilon^1 = C$ and $\varepsilon^2 = D$. Then

$$\begin{aligned}\Gamma^1_{22} = &\frac{5\,D\,C\,(C_{1.0} - D_{0.1})}{3\,N\,\Omega} + \frac{5\,D^2\,C_{0.1} - 5\,C^2\,D_{1.0}}{3\,N\,\Omega} + \\ &+ \frac{5\,P\,C^3 + 15\,Q\,C^2\,D + 15\,R\,C\,D^2 + 5\,S\,D^3}{3\,N\,\Omega}.\end{aligned} \tag{8.5}$$

Let's calculate the field K in special coordinates, where the coefficients of the equation (1.1) is defined by the formulas (6.2), (6.7), and (6.8):

$$K = -\frac{5}{48}\,\sigma(y)\,|x|^{-3/4} - \frac{5}{9} + \frac{9}{10}\,\frac{r(y)\,|x|^{-1/4}}{x} + \frac{7}{60}\,\frac{|x|^{-1/2}}{x} + \frac{6\,r(y)^2}{5\,x}.$$

Relying on this formula and on the relationships (6.13), we shall construct the following two pseudoscalar fields:

$$L = K\,N + \frac{5}{9}\,N + 3\,\Lambda\,\Omega + \frac{7}{9}\,\Omega^2 + 2\,\Lambda^2, \qquad I_2 = \frac{L^4}{N^2\,\Omega^4}. \tag{8.6}$$

Field L has the weight 2, while the weight of the field I_2 is zero, i. e. I_2 is a scalar invariant of the equation (1.1). Fields (8.6) can be evaluated explicitly:

$$L = \frac{25}{576}\,\frac{\sigma(y)\,|x|^{-3/4}}{x}, \qquad I_2 = \frac{15625}{2985984}\,\sigma(y)^4. \tag{8.7}$$

Second formula in (8.7) is similar to (6.15). Due to this formula we can make effective the second condition (6.9) from theorem 6.1:

$$I_2 = \text{const}\,. \tag{8.8}$$

Now we have only to make effective the rest third condition in (6.9). First, we evaluate the covariant derivatives of Λ and L along the pseudovectorial field ε. The field $\nabla_\varepsilon\Lambda$ has the weight 3, the weight of $\nabla_\varepsilon L$ is 4. Then we combine these two fields with the field (8.5), which has the weight 2:

$$\begin{aligned}E = \Gamma^1_{22} &- \frac{\nabla_\varepsilon L}{N} + \frac{4\,\Lambda\,\nabla_\varepsilon\Lambda}{N} + \frac{17\,\Omega\,\nabla_\varepsilon\Lambda}{6\,N} + \frac{12\,L^2}{5\,N} - \frac{53\,L\,\Lambda\,\Omega}{5\,N} - \\ &- \frac{48\,L\,\Lambda^2}{5\,N} - \frac{62\,L\,\Omega^2}{15\,N} - \frac{8\,L}{3} + \frac{48\,\Lambda^4}{5\,N} + \frac{106\,\Lambda^3\,\Omega}{5\,N} + \frac{16\,\Lambda^2}{3} + \\ &+ \frac{1163\,\Lambda^2\,\Omega^2}{60\,N} + \frac{137\,\Lambda\,\Omega^3}{18\,N} + \frac{50\,\Lambda\,\Omega}{9} + \frac{203\,\Omega^2}{108} + \frac{77\,\Omega^4}{135\,N} + \frac{20\,N}{27}.\end{aligned} \tag{8.9}$$

Field E in (8.9) has the weight 2. It is intentionally constructed so that its value in special coordinates is proportional to the function $s(y)$ from (6.8):

$$E = -\frac{s(y)\,|x|^{-1/2}}{64\,x^3}. \tag{8.10}$$

Due to this relationship (8.10) we can build the third scalar invariant for the second case of intermediate degeneration:

$$I_3=\frac{E^6\,N^4}{\Omega^{20}}. \tag{8.11}$$

The value of the invariant (8.11) in special coordinates doesn't depend on x:

$$I_3=\frac{625}{1296}\,s(y)^6.$$

This relationship is used to write the third condition (6.9) in invariant form:

$$I_3=\text{const}\,. \tag{8.12}$$

Now the theorem 6.1 can be reformulated as follows.

Theorem 8.1. *In the second case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if the conditions* (6.16), (8.8), *and* (8.12) *hold, i. e. if all invariants I_1, I_2, and I_3 are identically constant. If at least one of these conditions fails, then corresponding algebra of point symmetries is trivial.*

9. Third case of intermediate degeneration.

In third case of intermediate degeneration we have $F=0$. Parameters A and B do not vanish simultaneously. The conditions (6.1) are fulfilled too. But the condition (6.4) from the second case of intermediate degeneration is replaced by the following pair of relationship written in special coordinates:

$$c(y)=0,\qquad\qquad r(y)\neq 0. \tag{9.1}$$

First of these conditions is exactly opposite to (6.4). In effective form it is written as $\Omega=0$ (compare with the condition (6.6) above). By means of direct calculations for the field Λ we derive

$$\Lambda=-\frac{r(y)}{2\,x}. \tag{9.2}$$

Due to (9.2) the conditions (9.1) distinguishing third case of intermediate degeneration are written in the following invariant form:

$$\Omega=0,\qquad\qquad \Lambda\neq 0. \tag{9.3}$$

According to the results of [24], in this case one can find the special coordinates such that the coefficients P, Q, and R of the equation (1.1) are brought to the form

$$P=0,\qquad\qquad Q=-\frac{5}{12\,x},\qquad\qquad R=1. \tag{9.4}$$

The fourth coefficient S in the equation (1.1) is brought to the form

$$S=\sigma(y)\,|x|^{5/4}-4\,s(y)\,x+\frac{4}{3}\,|x|^2. \tag{9.5}$$

From (9.4) and (9.5) one can derive the following theorem (see [24]).

Theorem 9.1. *In the third case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if parameters* $s(y)$ *and* $\sigma(y)$ *in* (9.5) *are identically constant:*

$$s'(y)=0, \qquad \sigma'(y)=0. \tag{9.6}$$

If at least one of the conditions (9.6) *fails, then corresponding algebra of point symmetries is trivial.*

In order to make effective (9.6) let's consider again the field ω from (7.8). This field has the weight -1. Upon raising indices by means of the matrix (2.5) we get the vector field ω. In special coordinates its components are the following:

$$\omega^1=S_{1.0}-\frac{6}{5}\,R_{0.1}+\frac{12}{5}\,S\,Q-\frac{54}{25}\,R^2, \qquad \omega^2=\frac{6}{5}\,\Lambda. \tag{9.7}$$

Components of the vector-fields (9.7) in arbitrary coordinates can be effectively calculated by the formulas (7.12), (7.13), (7.14), and (7.15). One should only take into account that $\omega^1=\omega_2$ and $\omega^2=-\omega_1$.

From (9.7) we see that the condition of non-collinearity of the fields ω and α coincides with $\Lambda\neq 0$. In third case of intermediate degeneration this condition holds (see (9.3)). We shall use it to write the following expansions analogous to (8.3):

$$\begin{aligned}&\nabla_\alpha\alpha=\Gamma^1_{11}\,\alpha+\Gamma^2_{11}\,\omega, &&\nabla_\alpha\omega=\Gamma^1_{12}\,\alpha+\Gamma^2_{12}\,\omega,\\ &\nabla_\omega\alpha=\Gamma^1_{21}\,\alpha+\Gamma^2_{21}\,\omega, &&\nabla_\omega\omega=\Gamma^1_{22}\,\alpha+\Gamma^2_{22}\,\omega,\end{aligned} \tag{9.8}$$

The expansions (9.8) define the series of pseudoscalar fields: Γ^2_{11} is the field of the weight 4, fields Γ^1_{11}, Γ^2_{12}, and Γ^2_{21} have the weight 2, next three fields Γ^1_{12}, Γ^1_{21}, and Γ^2_{22} have the weight 0, and the last field Γ^1_{22} is of the weight -2. Most of these fields can be reduced to the various combinations of the fields that were already defined:

$$\Gamma^2_{11}=\Gamma^1_{21}=0, \qquad \Gamma^1_{11}=\Gamma^2_{21}=-\frac{3}{5}\,N.$$

The same is true for the next three fields too:

$$\Gamma^1_{12}=1-\frac{3}{5}\,K, \qquad \Gamma^2_{12}=\frac{9}{5}\,N, \qquad \Gamma^2_{22}=\frac{6}{5}\,K.$$

The only exception is the field Γ^1_{22}. In special coordinates this pseudoscalar field of the weight -2 can be calculated explicitly:

$$\begin{aligned}\Gamma^1_{22}&=S_{1.0}-\frac{6}{5}\,R_{0.1}-\frac{54}{25}\,R^2+\frac{12}{5}\,S\,Q+\frac{6}{5}\,\Lambda\,S_{1.1}-\\ &-\frac{36}{25}\,\Lambda\,R_{0.2}-\frac{36}{25}\,\Lambda^2\,S-\frac{3402}{625}\,R^3\,\Lambda-\frac{1026}{125}\,R_{0.1}\,R\,\Lambda+\\ &+\frac{72}{25}\,\Lambda\,S_{0.1}\,Q+\frac{63}{25}\,S_{1.0}\,R\,\Lambda-\frac{9}{5}\,S_{1.0}\,\Lambda_{0.1}+\frac{54}{25}\,R_{0.1}\,\Lambda_{0.1}+\\ &+\frac{486}{125}\,R^2\,\Lambda_{0.1}+\frac{1188}{125}\,S\,Q\,R\,\Lambda-\frac{108}{25}\,S\,Q\,\Lambda_{0.1}.\end{aligned} \tag{9.9}$$

Let's introduce the following notations:

$$C=\omega^1=\omega_2, \qquad\qquad D=\omega^2=-\omega_1. \tag{9.10}$$

Then for Γ^1_{22} we can derive the formula similar to (5.4) and (8.5):

$$\begin{aligned}\Gamma^1_{22}&=\frac{5\,C\,D\,(C_{1.0}-D_{0.1})}{6\,\Lambda}+\frac{5\,D^2\,C_{0.1}-5\,C^2\,D_{1.0}}{6\,\Lambda}+\\&+\frac{5\,P\,C^3+15\,Q\,C^2\,D+15\,R\,C\,D^2+5\,S\,D^3}{6\,\Lambda}.\end{aligned}\tag{9.11}$$

Formulas (9.10) and (9.11) together with (7.12), (7.13), (7.14), and (7.15) give the effective way to calculate the field (9.9) in arbitrary coordinates.

Now let's proceed with constructing the invariants for the equation (1.1) in the third case of intermediate degeneration. For L and I_1 in this case we take

$$L=K+\frac{5}{9}+\frac{2\,\Lambda^2}{N}, \qquad\qquad I_1=\frac{L^8\,N^6}{\Lambda^{12}}. \tag{9.12}$$

Both fields L and I_1 in (9.12) have the weigh 0, i. e. they are scalar invariants. For their values from (9.4) and (9.5) we derive

$$L=-\frac{5\,|x|^{-3/4}}{48}\,\sigma(y), \qquad\qquad I_1=\frac{6103515625}{20542695432781824}\,\sigma(y)^8. \tag{9.13}$$

Pseudoscalar field E having the weight -2 is composed of the fields ω, L, Ω, and N:

$$E=\Gamma^1_{22}-\frac{\nabla_{\omega}L}{N}+\frac{9\,L^2}{5\,N}-\frac{2\,L}{N}-\frac{12\,L\,\Lambda^2}{5\,N^2}+\frac{7\,\Lambda^2}{3\,N^2}+\frac{5}{9\,N}+\frac{63\,\Lambda^4}{20\,N^3}. \tag{9.14}$$

We shall use this field (9.14) in order to construct the second scalar invariant I_2:

$$I_2=\frac{E\,N^3}{\Lambda^4}. \tag{9.15}$$

Let's evaluate the fields E and I_2 using (9.4), (9.5), (9.14), and (9.15):

$$E=-\frac{36}{25\,x}\,s(y), \qquad\qquad I_2=\frac{5}{3}\,s(y). \tag{9.16}$$

Formulas (9.13) and (9.16) can be used to reformulate theorem 9.1 in effective form.

Theorem 9.2. *In the third case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if both invariants* I_1 *and* I_2 *in* (9.12) *and* (9.15) *are identically constant. If at least one of these two conditions fails, then corresponding algebra of point symmetries is trivial.*

10. Fourth case of intermediate degeneration.

In the fourth case of intermediate degeneration we have $F = 0$, and parameters A and B do not vanish simultaneously. Moreover, we have previous conditions $N \neq 0$ and $M = 0$ from (6.1), and also new ones

$$\Omega = 0, \qquad \Lambda = 0. \tag{10.1}$$

All above conditions, including (10.1), don't require special effectivization. They can be tested in arbitrary coordinates.

According to the results of [24], when the above conditions hold, one can choose special coordinates in which parameters P, Q, and R in (1.1) are brought to the form

$$P = 0, \qquad Q = -\frac{5}{12\,x}, \qquad R = 0. \tag{10.2}$$

The coefficient S in these coordinates also has the special form:

$$S = \sigma(y)\,|x|^{5/4} + \frac{4}{3}\,|x|^2. \tag{10.3}$$

Fourth case of intermediate degeneration is distinguished by the additional condition for the function $\sigma(y)$ in (10.3):

$$\sigma(y) \neq 0. \tag{10.4}$$

This condition (10.4) should be made effective. Fortunately in this case it doesn't require special efforts. One should only evaluate the field K in special coordinates, using the relationships (10.2) and (10.3):

$$K = -\frac{5\,|x|^{-3/4}}{48}\,\sigma(y) - \frac{5}{9}. \tag{10.5}$$

Due to (10.5) the condition (10.4) can be written as

$$K + \frac{5}{9} \neq 0. \tag{10.6}$$

Note that K is a scalar field, its weight is 0. Therefore, being fulfilled in special coordinates, condition (10.6) remains true for any other coordinates. Hence it is effective form for the condition (10.4).

Theorem 10.1. *In the fourth case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if the function* $\sigma(y)$ *from* (10.3) *is the solution of the following differential equation:*

$$\sigma''' - 5\,\frac{\sigma''\,\sigma'}{\sigma} + \frac{40}{9}\,\frac{{\sigma'}^3}{\sigma^2} = 0. \tag{10.7}$$

Otherwise if sigma(y) *doesn't satisfy the equation* (10.7), *then the algebra of point symmetries is trivial.*

This theorem from [24] gives complete description of the algebra of point symmetries for the fourth case of intermediate degeneration. But the condition (10.7) in its statement is written for the special coordinates. Now should make it effective. Let's consider pseudovectorial field ω from (7.8). This field was constructed by means of curvature tensor (7.1) for the case when $F = 0$ and $M = 0$. Under the conditions (10.1) its first component ω_1 equals to zero in special coordinates. This means that $\omega \parallel \alpha$, and we can define new scalar field Θ by the relationship $\omega = \Theta\,\alpha$. It is easy to find that Θ has the weight -2. Here is the expression for Θ in special coordinates:

$$\Theta = S_{1.0} - \frac{6}{5}\,R_{0.1} + \frac{12}{5}\,S\,Q - \frac{54}{25}\,R^2. \tag{10.8}$$

Fields K and Θ are bound with each other by very simple relationship: $K = N\,\Theta$. For the fourth case of intermediate degeneration $N \neq 0$, therefore Θ can be evaluated through K. However, when $N = 0$ (in sixth and seventh cases of intermediate degeneration), both fields K and N vanish simultaneously. For this reason it's better to determine Θ from the relationship $\omega = \Theta\,\alpha$. Due to this relationship we can make effective the formula (10.8) for Θ:

$$\Theta = \frac{\omega_2}{B}, \qquad\qquad \Theta = \frac{\omega_1}{A}. \tag{10.9}$$

First of the formulas (10.9) is used when $B \neq 0$, the value of ω_2 being calculated by (7.13). Second formula (10.9) is used for $A \neq 0$ when ω_1 is defined by (7.14).

Let's consider the covariant differential $\theta = \nabla\Theta$. This is pseudocovectorial field of the weight -2 with the following components:

$$\theta_1 = \Theta_{1.0} - 2\,\varphi_1\,\Theta, \qquad\qquad \theta_2 = \Theta_{0.1} - 2\,\varphi_2\,\Theta. \tag{10.10}$$

Formulas (10.9) and (10.10) define the field θ effectively in arbitrary coordinates. The quantities φ_1 and φ_2 in (10.10) should be calculated either by (4.8) or (4.14). In special coordinate we can calculate θ_1 and θ_2 explicitly:

$$\begin{aligned}\theta_1 = S_{1.1} - \frac{6}{5}\,R_{0.2} + \frac{6}{5}\,R\,S_{1.0} + \frac{12}{5}\,Q\,S_{0.1} +\\ + \frac{144}{25}\,S\,Q\,R - \frac{144}{25}\,R\,R_{0.1} - \frac{324}{125}\,R^3.\end{aligned} \tag{10.11}$$

$$\theta_2 = \frac{486}{125}\,Q\,R^2 - \frac{9}{5}\,Q\,S_{1.0} - \frac{108}{25}\,S\,Q^2 + \frac{54}{25}\,Q\,R_{0.1} - 1. \tag{10.12}$$

Let's raise indices in (10.11) and (10.12) by means of the matrix (2.5) and let's introduce the following notations:

$$\theta^1 = \theta_2 = C, \qquad\qquad \theta^2 = -\theta_1 = D. \tag{10.13}$$

The quantities θ^1 and θ^2 from (10.13) are components of the pseudovectorial field of the weight -1. Let's calculate its contraction with α:

$$L = -\frac{5}{9}\sum_{i=1}^{2}\alpha_i\,\theta^i = \frac{5}{9}\sum_{i=1}^{2}\theta_i\,\alpha^i. \tag{10.14}$$

Pseudoscalar field L from (10.14) has the weight 0, it is connected with the field K by the following relationship:

$$L = K + \frac{5}{9}. \tag{10.15}$$

Due to (10.15) and (10.6) in the fourth case of intermediate degeneration we have $L \neq 0$. For the field θ this means $\theta \nparallel \alpha$. Therefore we can consider the relationships

$$\begin{aligned} &\nabla_\alpha \alpha = \Gamma^1_{11}\,\alpha + \Gamma^2_{11}\,\theta, && \nabla_\alpha \theta = \Gamma^1_{12}\,\alpha + \Gamma^2_{12}\,\theta,\\ &\nabla_\theta \alpha = \Gamma^1_{21}\,\alpha + \Gamma^2_{21}\,\theta, && \nabla_\theta \theta = \Gamma^1_{22}\,\alpha + \Gamma^2_{22}\,\theta, \end{aligned} \tag{10.16}$$

which are similar to (5.2), (8.3), and (9.8). For seven coefficients in the expansions (10.16) we get the following relationships:

$$\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = 0, \qquad \Gamma^1_{11} = \Gamma^2_{21} = -\frac{3}{5}\,N, \qquad \Gamma^2_{12} = \frac{12}{5}\,N.$$

The formula for eighth coefficients Γ^1_{22} is quite different:

$$\begin{aligned} \Gamma^1_{22} = {} & \frac{9}{25}\,\Theta_{0.1}\,Q\,R\,\Theta + \frac{12}{5}\,Q\,\Theta^2_{0.1} - 3\,\Theta_{0.1}\,R - \frac{54}{125}\,R^2\,\Theta^2\,Q -\\ & -\frac{54}{25}\,R^2\,\Theta - \frac{9}{5}\,Q\,\Theta\,\Theta_{0.2} - \frac{54}{25}\,Q\,\Theta^2\,R_{0.1} - \Theta_{0.2} -\\ & -\frac{6}{5}\,R_{0.1}\,\Theta + \frac{81}{25}\,S\,Q^2\,\Theta^2 + \frac{18}{5}\,S\,Q\,\Theta + S. \end{aligned} \tag{10.17}$$

Formula (10.17) defines pseudoscalar field of the weight -4 in special coordinates. To make effective this formula let's use the notations (10.13) for the components of the field θ calculated by (10.10):

$$\begin{aligned} \Gamma^1_{22} = & -\frac{5\,D\,C\,(C_{1.0} - D_{0.1})}{9\,L} - \frac{5\,D^2\,C_{0.1} - 5\,C^2\,D_{1.0}}{9\,L} -\\ & -\frac{5\,P\,C^3 + 15\,Q\,C^2\,D + 15\,R\,C\,D^2 + 5\,S\,D^3}{9\,L}. \end{aligned} \tag{10.18}$$

Let's substitute the values of P, Q, R, and S from (10.2) and (10.3) into the above formulas (10.8) and (10.18) for pseudoscalar fields Θ and Γ^1_{22}. Then

$$\Theta = \frac{x\,|x|^{-3/4}}{4}\,\sigma(y) + \frac{4\,x}{3}. \tag{10.19}$$

The expression for Γ^1_{22} is more huge:

$$\begin{aligned} \Gamma^1_{22} = & \frac{|x|^{-3/2}}{64}\,\left(3\,\sigma''(y)\,\sigma(y) - 4\,\sigma'(y)^2\right) +\\ & + \frac{9\,|x|^{-1/4}}{256}\,\sigma(y)^3 + \frac{3\,x^2\,|x|^{-3/2}}{64}\,\sigma(y)^2. \end{aligned} \tag{10.20}$$

Now let's return to the differential equation (10.7), which we are to make effective. It can be written in the form that doesn't contain third order derivatives:

$$\frac{\bigl(3\,\sigma''(y)\,\sigma(y)-4\,\sigma'(y)^2\bigr)^3}{\sigma(y)^{10}}=\text{const}\,. \tag{10.21}$$

By means of (10.19) and (10.20) we compose one more pseudoscalar field:

$$E=\Gamma^1_{22}+\frac{27\,N}{5}\left(\Theta+\frac{5}{9\,N}\right)^3-\frac{3}{4}\left(\Theta+\frac{5}{9\,N}\right)^2. \tag{10.22}$$

Field E in (10.22) has the weight -4. We use it to construct first scalar invariant for the fifth case of intermediate degeneration:

$$I_1=\frac{E^6}{N^8}\left(\Theta+\frac{5}{9\,N}\right)^{-20}=\frac{E^6\,N^{12}}{L^{20}}. \tag{10.23}$$

Using (10.19) and (10.20) we can calculate this invariant (10.23) in explicit form:

$$I_1=\frac{6879707136}{390625}\cdot\frac{\bigl(3\,\sigma''(y)\,\sigma(y)-4\,\sigma'(y)^2\bigr)^6}{\sigma(y)^{20}}. \tag{10.24}$$

Comparing formulas (10.24) and (10.21), we can reformulate theorem 10.1 in the form that admit effective testing in arbitrary coordinates.

Theorem 10.2. *In the fourth case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if the invariant* (10.23) *is identically constant. Otherwise this algebra is trivial.*

11. Fifth case of intermediate degeneration.

In the fifth case of intermediate degeneration we have the condition $F=0$, which is common for all cases with degeneration. Parameters A and B do not vanish simultaneously. As in the fourth case, here we have the conditions (6.1) and (10.1). But the condition (10.4) is replaced by quite opposite condition $\sigma(y)=0$ (see [24]). It can be easily written in effective form:

$$K+\frac{5}{9}=0. \tag{11.1}$$

Here the field K of the weight 0 is calculated either by (7.22) or (7.23). According to the results of paper [24], if all above conditions, including (11.1), are fulfilled, then the equation (1.1) can be brought to the form

$$y''=-\frac{5}{12\,x}\,y'+\frac{4}{3}\,x^2\,{y'}^3 \tag{11.2}$$

in some special coordinates. This equation (11.2) doesn't contain arbitrary parameters. Therefore its algebra of point symmetries is quite definite. According to [24], it is three-dimensional and isomorphic to the matrix algebra $\mathrm{sl}(2,\mathbb{R})$.

12. Sixth case of intermediate degeneration.

Second, third, fourth and fifth cases of intermediate degeneration are united by the fact that in all these cases we have $M = 0$ and $N \neq 0$ (see conditions (6.1)). Sixth an seventh cases are separate in the sense of these conditions. Here they are replaced by the following one:

$$N = 0. \tag{12.1}$$

The relationship $M = 0$ is now derived from (12.1) and $F = 0$. Parameters A and B do not vanish simultaneously in sixth and seventh cases too.

Suppose that all above conditions are fulfilled. Let's transform the equation (1.1) to the special coordinates defined by the conditions (1.8). Then, according to the results of [24], for P and Q we get

$$P = 0, \qquad\qquad Q = 0. \tag{12.2}$$

Coefficient R in such coordinates is given by the relationship

$$R = c(y)\,x + r(y). \tag{12.3}$$

In special coordinates the sixth case of intermediate degeneration is distinguished by the additional condition written in terms of the function $c(y)$ from (12.3):

$$c(y) \neq 0. \tag{12.4}$$

The condition (12.4), which is written in special coordinates, should be made effective by transformation to the arbitrary coordinates. For this purpose we shall use the fact that from $F = 0$ and $N = 0$ we have $M = 0$. This means that in sixth and seventh cases of intermediate degeneration the pseudoscalar fields Ω and Λ are defined, as well as the fields constructed by means of curvature tensor in section 7. Let's evaluate them in special coordinates:

$$\Omega = c(y), \qquad\qquad \Lambda = -2\,c(y). \tag{12.5}$$

Formulas (12.5) give us the required effectivization for the condition (12.4):

$$\Omega \neq 0. \tag{12.6}$$

From (12.5) and (12.6) we get equivalent condition $\Lambda \neq 0$. Moreover, in sixth case of intermediate degeneration we have the following relationship binding Ω and Λ:

$$\Lambda = -2\,\Omega. \tag{12.7}$$

Let's substitute $Q = 0$ from (12.2) and $\Lambda = -2\,\Omega$ from (12.7) into the formula (7.8) for the components of the field ω in special coordinates. This gives us

$$\omega_1 = \frac{9}{5}\,\Omega, \qquad\qquad \omega_2 = S_{1.0} - \frac{6}{5}\,R_{0.1} - \frac{54}{25}\,R^2. \tag{12.8}$$

Let's raise indices and let's introduce the following notations:

$$C=\omega^1=\omega_2, \qquad\qquad D=\omega^2=-\omega_1. \tag{12.9}$$

From (12.8) and (12.6) we see that ω and α are non-collinear. Therefore we can consider the expansions, which coincides with (9.8):

$$\begin{aligned}
&\nabla_\alpha\alpha=\Gamma^1_{11}\,\alpha+\Gamma^2_{11}\,\omega, &&\nabla_\alpha\omega=\Gamma^1_{12}\,\alpha+\Gamma^2_{12}\,\omega,\\
&\nabla_\omega\alpha=\Gamma^1_{21}\,\alpha+\Gamma^2_{21}\,\omega, &&\nabla_\omega\omega=\Gamma^1_{22}\,\alpha+\Gamma^2_{22}\,\omega,
\end{aligned} \tag{12.10}$$

But for the coefficients of the expansions (12.10) in the sixth case of intermediate degeneration we get the formulas, which are different from that of the third case:

$$\begin{aligned}
&\Gamma^1_{11}=\Gamma^2_{11}=\Gamma^2_{12}=0, &&\Gamma^1_{21}=\Gamma^2_{21}=0,\\
&\Gamma^2_{12}=1-\frac{12}{25}\,K, &&\Gamma^2_{22}=\frac{27}{25}\,K.
\end{aligned}$$

The formula for Γ^1_{22} is written in special coordinates:

$$\begin{aligned}
\Gamma^1_{22}=&-\frac{93}{25}\,S_{1.0}\,R\,\Omega+\frac{13}{5}\,S_{1.0}\,\Omega_{0.1}+S_{1.0}+\frac{306}{25}\,R_{0.1}\,R\,\Omega-\\
&-\frac{78}{25}\,R_{0.1}\,\Omega_{0.1}-\frac{6}{5}\,R_{0.1}+\frac{5022}{625}\,R^3\,\Omega-\frac{702}{125}\,R^2\,\Omega_{0.1}-\\
&-\frac{54}{25}\,R^2-\frac{9}{5}\,\Omega\,S_{1.1}+\frac{54}{25}\,\Omega\,R_{0.2}+\frac{81}{25}\,\Omega^2\,S.
\end{aligned} \tag{12.11}$$

In arbitrary coordinates we should replace (12.11) by

$$\begin{aligned}
\Gamma^1_{22}=&-\frac{5\,C\,D\,(C_{1.0}-D_{0.1})}{9\,\Omega}-\frac{5\,D^2\,C_{0.1}-5\,C^2\,D_{1.0}}{9\,\Omega}-\\
&-\frac{5\,P\,C^3+15\,Q\,C^2\,D+15\,R\,C\,D^2+5\,S\,D^3}{9\,\Omega}.
\end{aligned} \tag{12.12}$$

Parameters C and D in (12.12) introduced by (12.9) should be calculated by (7.12) and (7.13) for $B\neq 0$, or by (7.14) and (7.15) for $A\neq 0$.

In [24] it was shown that at the expense of further specialization of the choice of coordinates in sixth case of intermediate degeneration coefficients R and S in the equation (1.1) can be brought to the form

$$R=x, \qquad\qquad S=x^3+\frac{1}{2}\,x^2+\sigma(y)\,x+s(y). \tag{12.13}$$

There also the following theorem was proved.

Theorem 12.1. *In the sixth case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if the functions* $\sigma(y)$ *and* $s(y)$ *in* (12.13) *are identically constant:*

$$s'(y)=0, \qquad\qquad \sigma'(y)=0. \tag{12.14}$$

If at least one of the conditions (12.14) *fails, then corresponding algebra of point symmetries is trivial.*

Let's use (12.2) and (12.13) to evaluate Ω and Λ. Their values in special coordinates appear to be constant:

$$\Omega = 1, \qquad \Lambda = -2. \tag{12.15}$$

The relationships (12.15) can be derived from (12.5) and (12.13). Let's also evaluate the field K of the weight 0 and its covariant derivative along ω:

$$K = x, \qquad \nabla_\omega K = \frac{21}{25}\,x^2 + x + \sigma(y). \tag{12.16}$$

Using (12.16), we construct another field L of the weight 0 by means of K and $\nabla_\omega K$. We shall choose this field as the first scalar invariant:

$$I_1 = L = \nabla_\omega K - \frac{21}{25}\,K - K. \tag{12.17}$$

The field (12.17) is constructed so that its value in special coordinates coincides with the value of the function $\sigma(y)$ in (12.13):

$$I_1 = L = \sigma(y). \tag{12.18}$$

It's not difficult to evaluate the covariant derivative of L along the vector-field ω:

$$\nabla_\omega L = -\frac{9}{5}\,\sigma'(y). \tag{12.19}$$

Now we shall use (12.18), (12.19), and pseudoscalar field Γ^1_{22} of the weight -2 from (12.10) in order to construct the second scalar invariant:

$$I_2 = \Omega^2\,\Gamma^1_{22} - \nabla_\omega L - \frac{72}{625}\,K^3 + \frac{63}{50}\,K^2 + \frac{12}{25}\,K\,L - K - L. \tag{12.20}$$

The invariant (12.20) is constructed so that its value in special coordinates is proportional to the function $s(y)$ in (12.3):

$$I_2 = \frac{81}{25}\,s(y). \tag{12.21}$$

From (12.18) and (12.21) we derive the following effectivization for the theorem 12.1

Theorem 12.2. *In the sixth case of intermediate degeneration algebra of point symmetries of the equation* (1.1) *is one-dimensional if and only if both invariants* I_1 *and* I_2 *from* (12.17) *and* (12.20) *are identically constant. Otherwise this algebra of point symmetries is trivial.*

13. Seventh case of intermediate degeneration.

In seventh case of intermediate degeneration the conditions $F = 0$ and $N = 0$ remains the same as in sixth case. They give $M = 0$ as immediate consequence. Parameters A and B do not vanish simultaneously. However, the condition $c(y) \neq 0$ from (12.4) is replaced by opposite one: $c(y) = 0$. In effective form this condition is written as the condition opposite to (12.6):

$$\Omega = 0. \tag{13.1}$$

From (13.1) and (12.7) we get $\Lambda = 0$. From (12.8) we find that in special coordinates $\omega_1 = 0$. This means that the fields ω and α are collinear, hence the expansions like (12.10) in this case are impossible. But at the same time the collinearity of ω and α is exactly the condition for existence of the field Θ in (10.9). In special coordinates this field is calculated by (10.8). Repeating the constructions from the section 10, we define pseudocovectorial field θ of the weight -2 with components (10.10). Thereafter we raise indices in accordance with (10.13) and we obtain the pseudovectorial field θ of the weight -1. Formulas (10.11) and (10.12) for the components of θ in special coordinates in seventh case of intermediate degeneration are substantially more simple:

$$\begin{aligned} &\theta^1 = S_{1.1} - \frac{6}{5}\,R_{0.2} - \frac{144}{25}\,R\,R_{0.1} + \frac{6}{5}\,R\,S_{1.0} - \frac{324}{125}\,R^3,\\ &\theta^2 = -1. \end{aligned} \tag{13.2}$$

From (13.2) we see, that $\theta \nparallel \alpha$. Therefore we can consider the expansions

$$\begin{aligned} &\nabla_\alpha\alpha = \Gamma^1_{11}\,\alpha + \Gamma^2_{11}\,\theta, &&\nabla_\alpha\theta = \Gamma^1_{12}\,\alpha + \Gamma^2_{12}\,\theta,\\ &\nabla_\theta\alpha = \Gamma^1_{21}\,\alpha + \Gamma^2_{21}\,\theta, &&\nabla_\theta\theta = \Gamma^1_{22}\,\alpha + \Gamma^2_{22}\,\theta, \end{aligned} \tag{13.3}$$

which reproduce (10.16) for the seventh case of intermediate degeneration. Almost all coefficients in (13.3) appear to be zero. Exception is the coefficient Γ^1_{22}, which defines pseudoscalar field of the weight -4. In special coordinates it is calculated by

$$\Gamma^1_{22} = S - \Theta_{0.2} - \frac{6}{5}\,R_{0.1}\,\Theta - 3\,R\,\Theta_{0.1} - \frac{54}{25}\,R^2\,\Theta. \tag{13.4}$$

In order to recalculate (13.4) in arbitrary coordinates let's denote by C and D the components of the field θ as it was done in (10.13). They should be calculated by (10.10). Then for Γ^1_{22} we get

$$\begin{aligned} \Gamma^1_{22} = C\,D\,(D_{0.1} - C_{1.0}) + C^2\,D_{1.0} - D^2\,C_{0.1} -\\ - P\,C^3 - 3\,Q\,C^2\,D - 3\,R\,C\,D^2 - S\,D^3. \end{aligned} \tag{13.5}$$

Formula (13.5) has no denominator in right hand side, since $\Omega = -\theta^1\,\alpha_1 - \theta^2\,\alpha_2 = 1$.

Now let's use the results of the paper [24]. There it was shown that in the seventh case of intermediate degeneration upon special choice of coordinates the coefficients of the equation (1.1) can be brought to the form

$$\begin{aligned} &P = 0, &&Q = 0,\\ &R = 0, &&S = \frac{1}{2}\,x^2 + s(y). \end{aligned} \tag{13.6}$$

The conditions (1.8) for these coordinates appear to be fulfilled. The structure of algebra of point symmetries for this case is determined by the function $s(y)$ in (13.6). Here we have two theorems from [24].

Theorem 13.1. *In the seventh case of intermediate degeneration the algebra of point transformations of the equation* (1.1) *is two-dimensional if and only if the function* $s(y)$ *in* (13.6) *is identically zero. This algebra is integrable but it is not Abelian.*

Theorem 13.2. *In the seventh case of intermediate degeneration the algebra of point transformations of the equation* (1.1) *is one-dimensional if and only if the function* $s(y)$ *in* (13.6) *is nonzero and if this function is the solution of the following differential equation:*

$$4\,s''(y) - \frac{5\,s'(y)^2}{s(y)} = 0. \tag{13.7}$$

If $s(y)$ *doesn't satisfy the differential equation* (13.7)*, then the algebra of point symmetries is trivial.*

Let's substitute (13.6) into the formula (10.8) for the field Θ, into the formulas (13.2) for θ, and into the formula (13.4) for the field Γ^1_{22}. Then we get

$$\Theta = x, \qquad\qquad \Gamma^1_{22} = \frac{1}{2}x^2 + s(y). \tag{13.8}$$

The components of the field θ appear to be unitary:

$$\theta^1 = 0, \qquad\qquad \theta^2 = -1. \tag{13.9}$$

The field Γ^1_{22} in (13.8) has the weight -4, and the field Θ has the weight -2. Therefore the following field L has the weight -4:

$$L = \Gamma^1_{22} - \frac{1}{2}\,\Theta^2. \tag{13.10}$$

It is easy to evaluate the field (13.10) in special coordinates:

$$L = s(y). \tag{13.11}$$

Formula (13.11) yields the effectivization for the above theorem 13.1.

Theorem 13.3. *In the seventh case of intermediate degeneration the algebra of point transformations of the equation* (1.1) *is two-dimensional if and only if the pseudoscalar field* L *in* (13.10) *is identically zero. This algebra is integrable but it is not Abelian.*

In order to make effective the theorem 13.2, first, we note that for $s(y) \neq 0$ the equation (13.7) can be written as

$$\frac{s'(y)^4}{s(y)^5} = \text{const}\,. \tag{13.12}$$

Then we calculate the covariant derivative of the field L along the field θ. The resulting field $\nabla_\theta L$ has the weight -5. Due to (13.9) its value is given by

$$\nabla_\theta L = -s'(y). \tag{13.13}$$

Now we are able to compose the scalar invariant I_1 by $\nabla_\theta L$ and L:

$$I_1 = \frac{(\nabla_\theta L)^4}{L^5}. \tag{13.14}$$

Weight of tie field (13.14) is equal to zero. Due to (13.11) and (13.13) the value of this field coincides with the left hand side of (13.12).

Theorem 13.4. *In the seventh case of intermediate degeneration the algebra of point transformations of the equation* (1.1) *is one-dimensional if and only if the field* L *in* (13.10) *is nonzero and if the invariant* I_1 *in* (13.14) *is identically constant.*

If the first hypothesis in the theorem 13.4 is not fulfilled, i. e. if $L = 0$, then the algebra of point symmetries of the equation (1.1) is two-dimensional due to the theorem 13.3. If $L = 0$, but $I_1 \neq \text{const}$, then this algebra is trivial due to the theorem 13.2.

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