ON A POINT SYMMETRY ANALYSIS FOR GENERALIZED DIFFUSION TYPE EQUATIONS.

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Abstract. Generalized diffusion type equations are considered and point symmetry analysis is applied to them. The equations with extremal order point symmetry algebras are described. Some old geometrical results are rederived in connection with theory of these equation.

1. Introduction.

In this paper we consider a class of systems of evolutional equations, which we call *the equations of generalized diffusion*:

$$\frac{\partial y^i}{\partial t}=\sum_{j=1}^{n}A^i_j\left(\frac{\partial^2 y^j}{\partial x^2}+\sum_{r=1}^{n}\sum_{s=1}^{n}\Gamma^j_{rs}\,\frac{\partial y^r}{\partial x}\,\frac{\partial y^s}{\partial x}\right),\quad i=1,\,\ldots\,,\,n. \tag{1.1}$$

Here x and t are independent variables, $y^1,\,\ldots\,,\,y^n$ are dependent variables, and coefficients A^i_j and Γ^j_{rs} are the functions of $y^1,\,\ldots\,,\,y^n$. They do not depend on x and t explicitly.

Special case of (1.1) is formed by *the equations of diffusional type*, which describe the diffusion phenomena in multicomponent mixtures:

$$\frac{\partial y^i}{\partial t}=\sum_{j=1}^{n}\frac{\partial}{\partial x}\left(A^i_j\,\frac{\partial y^j}{\partial x}\right),\quad i=1,\,\ldots\,,\,n, \tag{1.2}$$

(see paper [1] and reference list there). The equations of magnet in Heisenberg model (see [2] and [3]) are also reduced to (1.1) in classical limit:

$$\mathbf{S}_t=[\mathbf{S},\,\mathbf{S}_{xx}]. \tag{1.3}$$

Here $\mathbf{S}$ is a unit length three-dimensional vector, which describes the magnetization of the media. Square brackets denote the vector product. The equations (1.3) have multidimensional generalization, where $\mathbf{S}$ is an element of some Lie algebra and

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square brackets denote the commutator in this algebra (see for instance [4] and [5]). They can be restricted to any orbit of coadjoint action of corresponding Lie group G on L. When written in local coordinates $y^1, \dots, y^n$ on such orbit, the equations (1.3) have the form (1.1)

An important feature of the class of equations (1.1) is that it is closed with respect to the point transformations of the form

$$\begin{aligned} &\tilde{y}^1 = \tilde{y}^1(y^1, \dots, y^n),\\ &\dots\dots\dots\dots\dots\\ &\tilde{y}^n = \tilde{y}^1(y^1, \dots, y^n), \end{aligned} \tag{1.4}$$

which do not change independent variables x and t. Therefore we can treat variables $y^1, \dots, y^n$ as local coordinates on some manifold[1]. Let's denote it by M. Change of variables (1.4) corresponds to the transition from one local map on M to another. By means of direct calculations one derives the following transformation rules for A^i_j and Γ^j_{rs} under the point transformation of the form (1.4):

$$A^k_i = \sum_{m=1}^{n}\sum_{p=1}^{n} S^k_m\, T^p_i\, \tilde{A}^m_p, \tag{1.5}$$

$$\Gamma^k_{ij} = \sum_{m=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} S^k_m\, T^p_i\, T^q_j\, \tilde{\Gamma}^m_{pq} + \sum_{m=1}^{n} S^k_m\, \frac{\partial T^m_i}{\partial y^j}. \tag{1.6}$$

Here T and S are Jacobi matrices for direct and inverse transformation (1.4):

$$S^i_j = \frac{\partial y^i}{\partial \tilde{y}^j}, \qquad\qquad T^i_j = \frac{\partial \tilde{y}^i}{\partial y^j}.$$

In geometry it is customary to call S *the matrix of direct transition*, while T is called *the inverse transition matrix* (see for instance [6]).

From (1.5) and (1.6) one can understand that parameters Γ^j_{rs} define symmetric affine connection Γ on M and parameters A^i_j define a tensor field $\mathbf{A}$ of the type $(1,1)$. This provides us with ample opportunity to apply powerful differential geometric methods to the study of equations (1.1), e. g. in [1] we have found an effective criterion for checking whether the equations (1.1) can be brought to the form (1.2) by means of some transformation (1.4) or not. In this paper we consider the problem of describing the equations (1.1) whose point symmetry algebras are large enough (extremal in some sense).

2. Vector fields and point symmetries.

Let $\boldsymbol{\eta}$ be a vector field on the manifold M. In local coordinates it is represented by differential operator

$$\boldsymbol{\eta} = \sum_{i=1}^{n} \eta^i\, \frac{\partial}{\partial y^i}. \tag{2.1}$$

[1] for equations (1.3) such manifold comes from initial statement of the problem — this is the orbit of coadjoint action of Lie group on its Lie algebra.

Vector field (2.1) gives rise to the local one-parametric group of transformations $\varphi_\tau : M \to M$, while each solution of (1.1) is interpreted as two-parametric set of points in M (x and t are parameters). Applying φ_τ to this set we obtain another two-parametric set of points with the same parameters x and t. Vector field $\boldsymbol{\eta}$ is called *a point symmetry* for some particular system of equations, if for any solution of this system and for arbitrary value of τ the transformed two-parametric set of points satisfies the same equations (1.1) as the initial set. From this statement one can derive the following determining equations for the field of point symmetry:

$$\sum_{k=1}^{n}\eta^k\,\frac{\partial A^i_j}{\partial y^k}-\sum_{k=1}^{n}A^k_j\,\frac{\partial\eta^i}{\partial y^k}+\sum_{k=1}^{n}A^i_k\,\frac{\partial\eta^k}{\partial y^j}=0, \tag{2.2}$$

$$\begin{gathered}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial\eta^i}{\partial y^k}\,A^k_j-\frac{\partial A^i_j}{\partial y^k}\,\eta^k\right)\Gamma^j_{rs}=\\=\sum_{j=1}^{n}\sum_{k=1}^{n}A^i_j\left(\frac{\partial^2\eta^j}{\partial y^r\,\partial y^s}+\frac{\partial\Gamma^j_{rs}}{\partial y^k}\,\eta^k+\Gamma^j_{ks}\,\frac{\partial\eta^k}{\partial y^r}+\Gamma^j_{rk}\,\frac{\partial\eta^k}{\partial y^s}\right).\end{gathered} \tag{2.3}$$

Note that there is an advanced theory of point symmetries for the systems of differential equations (see [7] or [8]). Application of this theory leads to the same determining equations (2.2) and (2.3) for $\boldsymbol{\eta}$. Note also that (2.1) is a special form of point symmetry for equations (1.1). Generic point symmetry has the form

$$\boldsymbol{\eta}=\theta\,\frac{\partial}{\partial t}+\xi\,\frac{\partial}{\partial x}+\sum_{i=1}^{n}\eta^i\,\frac{\partial}{\partial y^i}. \tag{2.4}$$

In this paper we restrict our consideration to the special point symmetries (2.1) since they have transparent geometric interpretation as vector fields on the manifold M. Geometric interpretation of generic point symmetries (2.4) is the subject for separate paper.

Determining equations (2.2) and (2.3) for the field of point symmetry (2.1) can be written without reference to local coordinates. First of them provides vanishing of Lie derivative of tensor field $\mathbf{A}$ (see more details in [9]):

$$L_{\boldsymbol{\eta}}(\mathbf{A})=0. \tag{2.5}$$

In case of $\det\mathbf{A}\neq 0$ from (2.2) and (2.3) we derive the equation

$$\sum_{k=1}^{n}\frac{\partial\eta^i}{\partial y^k}\,\Gamma^k_{rs}=\frac{\partial^2\eta^i}{\partial y^r\,\partial y^s}+\sum_{k=1}^{n}\left(\frac{\partial\Gamma^i_{rs}}{\partial y^k}\,\eta^k+\Gamma^i_{ks}\,\frac{\partial\eta^k}{\partial y^r}+\Gamma^i_{rk}\,\frac{\partial\eta^k}{\partial y^s}\right) \tag{2.6}$$

that can replace (2.3). If we take into account symmetry of connection components $\Gamma^i_{rs}=\Gamma^i_{sr}$, which follows from (1.1), we can simplify this equation as below:

$$\nabla_r\nabla_s\eta^i=\sum_{k=1}^{n}R^i_{srk}\,\eta^k. \tag{2.7}$$

Here R^i_{srk} are the components of curvature tensor, defined by the formula

$$R^i_{srk} = \frac{\partial \Gamma^i_{ks}}{\partial y^r} - \frac{\partial \Gamma^i_{rs}}{\partial y^k} + \sum_{q=1}^{n} \Gamma^q_{ks}\,\Gamma^i_{rq} - \sum_{q=1}^{n} \Gamma^q_{rs}\,\Gamma^i_{kq} \tag{2.8}$$

(see for instance [6]). In invariant (coordinateless) form the equations (2.7) are written as the relationship determining the commutator of Lie derivative $L_{\boldsymbol{\eta}}$ with covariant derivative along an arbitrary vector field $\mathbf{Y}$:

$$[L_{\boldsymbol{\eta}},\, \nabla_{\mathbf{Y}}] = \nabla_{[\boldsymbol{\eta},\,\mathbf{Y}]}. \tag{2.9}$$

Here one can remember the following lemma from [10].

Lemma 2.1. *Vector field $\boldsymbol{\eta}$ is an infinitesimal affine transformation*[1] *on the manifold M if and only if the relationship*

$$[L_{\boldsymbol{\eta}},\, \nabla_{\mathbf{Y}}]\mathbf{Z} = \nabla_{[\boldsymbol{\eta},\,\mathbf{Y}]}\mathbf{Z} \tag{2.10}$$

holds for two arbitrary vector fields $\mathbf{Y}$ and $\mathbf{Z}$ on this manifold.

The relationship $[L_{\boldsymbol{\eta}},\, \nabla_{\mathbf{Y}}]\varphi = \nabla_{[\boldsymbol{\eta},\,\mathbf{Y}]}\varphi$ for scalar field φ holds identically. Therefore from (2.10) we obtain the relationship

$$[L_{\boldsymbol{\eta}},\, \nabla_{\mathbf{Y}}]\mathbf{W} = \nabla_{[\boldsymbol{\eta},\,\mathbf{Y}]}\mathbf{W} \tag{2.11}$$

which holds for arbitrary tensor field $\mathbf{W}$. It is equivalent to (2.9). We state this result as a theorem.

Theorem 2.1. *Vector field $\boldsymbol{\eta}$ is a field of point symmetry for the system of equations* (1.1) *with non-degenerate matrix A if and only if it is an infinitesimal affine transformation for symmetric affine connection with components Γ^j_{rs} and if Lie derivative of $\mathbf{A}$ along $\boldsymbol{\eta}$ is equal to zero (i. e. both relationships* (2.5) *and* (2.9) *hold simultaneously).*

3. Operator field A and tensor fields relative to it.

According to the above geometric interpretation the system of equations (1.1) describes the dynamics in x and t for points of some manifold M equipped with an affine connection Γ and with operator field $\mathbf{A}$. Due to (2.5) operator field $\mathbf{A}$ is invariant with respect to the field of point symmetry $\boldsymbol{\eta}$. From (2.5) one can derive invariance for all integer powers of the operator field $\mathbf{A}$. This means

$$L_{\boldsymbol{\eta}}(\mathbf{A}^q) = 0, \text{ where } q \in \mathbb{Z}. \tag{3.1}$$

We suppose operator field $\mathbf{A}$ to be non-degenerate, therefore we admit negative integer powers $q < 0$ in (3.1).

[1] here the term "affine transformation" is understood as a map preserving an affine connection (an automorphism of an affine connection).

Besides with fields $\mathbf{A}^q$ the operator field $\mathbf{A}$ gives rise to a set of tensor fields of type $(1, m)$, where $m \geqslant 1$ is a positive integer number. They are completely skew-symmetric in lower indices. These fields are generated by integer powers of $\mathbf{A}$ and Froelicher-Nijenhuis bracket from [11]. Froelicher-Nijenhuis bracket plays an important role in the theory of equations of hydrodynamical type

$$\frac{\partial y^i}{\partial t} = \sum_{j=1}^{n} A^i_j \, \frac{\partial y^j}{\partial x}, \quad i = 1, \ldots, n, \tag{3.2}$$

where manifold M has no connection and the only differential geometric object on it is the operator field $\mathbf{A}$. For instance in [12] we used Froelicher-Nijenhuis bracket for to obtain tensorial form $\mathbf{P} = 0$ of integrability condition for the equations (3.2) integrable by means of generalized hodograph method from [13]. Components of tensor field $\mathbf{P}$ were expressed through components of matrix A. Therefore $\mathbf{P} = 0$ appears to be an effective integrability test, which do not require the calculation of Riemann invariants required by the theory from [13]. Effective procedure for point classification in the class of ordinary differential equations of the form

$$y'' = P(x, y) + 3\, Q(x, y)\, y' + 3\, R(x, y)\, {y'}^2 + S(x, y)\, {y'}^3$$

were derived in paper [14]. However, this was done without use of Froelicher-Nijenhuis bracket.

Let's consider tensor field $\mathbf{B}$ of type $(1, p)$ and tensor field $\mathbf{C}$ of type $(1, q)$ made up by two vector fields $\mathbf{b}$ and $\mathbf{c}$ and two differential forms:

$$\mathbf{B} = \mathbf{b} \otimes \beta, \qquad\qquad \mathbf{C} = \mathbf{c} \otimes \gamma. \tag{3.3}$$

For the fields (3.3) Froelicher-Nijenhuis bracket is calculated as follows:

$$\begin{aligned} \{\mathbf{B}, \mathbf{C}\} = [\mathbf{b},\, \mathbf{c}] \otimes \beta \wedge \gamma - \mathbf{b} \otimes L_{\mathbf{c}}\beta \wedge \gamma + \mathbf{c} \otimes \beta \wedge L_{\mathbf{b}}\gamma + \\ + (-1)^p\, \mathbf{b} \otimes \iota_{\mathbf{c}}\beta \wedge d\gamma + (-1)^p\, \mathbf{c} \otimes d\beta \wedge \iota_{\mathbf{b}}\gamma. \end{aligned} \tag{3.4}$$

Here d is an external differentiation defined by the formula

$$\begin{aligned} d\omega(\mathbf{X}_0, \ldots, \mathbf{X}_r) = \sum_{i=0}^{r} \frac{(-1)^i}{r+1}\, \mathbf{X}_i(\omega(\mathbf{X}_0, \ldots, \hat{\mathbf{X}}_i, \ldots, \mathbf{X}_r)) + \\ + \sum\sum_{0 \leqslant i < j \leqslant r} \frac{(-1)^{i+j}}{r+1}\, \omega([\mathbf{X}_i,\, \mathbf{X}_j], \mathbf{X}_0, \ldots, \hat{\mathbf{X}}_i, \ldots, \hat{\mathbf{X}}_j, \ldots, \mathbf{X}_r), \end{aligned} \tag{3.5}$$

where ω is a differential r-form and $\mathbf{X}_0, \ldots, \mathbf{X}_r$ are arbitrary vectorial fields. Hat over the sign of vector field means that field with this particular number is omitted from the list of arguments of the form ω. $L_{\mathbf{b}}$ and $L_{\mathbf{c}}$ in (3.4) are Lie derivatives along vector fields $\mathbf{b}$ and $\mathbf{c}$, while $\iota_{\mathbf{b}}$ and $\iota_{\mathbf{c}}$ are the differentiations of substitution. For r-form ω and vector field $\mathbf{c}$ the expression $\iota_{\mathbf{c}}\omega$ denotes $(r-1)$-form such that

$$\iota_{\mathbf{c}}\omega(\mathbf{X}_1, \ldots, \mathbf{X}_{r-1}) = r\, \omega(\mathbf{c}, \mathbf{X}_1, \ldots, \mathbf{X}_{r-1}). \tag{3.6}$$

The operation $\iota_{\mathbf{c}}$ is also known as *internal product* with respect to the vector field $\mathbf{c}$ (see [9]). For the sake of completeness we give the formula that define external product $\beta\wedge\gamma$ as a result of alternation of $\beta\otimes\gamma$:

$$(\beta\wedge\gamma)(\mathbf{X}_1,\ldots,\mathbf{X}_{p+q})=\sum_{\sigma}\frac{(-1)^{\sigma}}{(p+q)!}\,(\beta\otimes\gamma)(\mathbf{X}_{\sigma_1},\ldots,\mathbf{X}_{\sigma(p+q)}). \tag{3.7}$$

Here σ is a permutation running over the whole group of permutations of the order $p+q$, while $(-1)^{\sigma}$ is a sign coefficient determined by the parity of σ.

Theorem 3.1. *Froelicher-Nijenhuis bracket* $\{\mathbf{B},\mathbf{C}\}$ *defined in* (3.4) *for tensor fields* (3.3) *can be expanded to the case of arbitrary two skew-symmetric tensor fields of types* $(1,p)$ *and* $(1,q)$.

Proof of the theorem 3.1 based on formulas (3.4), (3.5), (3.6), and (3.7) can be found in [12]. On the base of the same formulas one can check the following relationships:

$$\begin{aligned}&\{\mathbf{B},\mathbf{C}\}+(-1)^{pq}\,\{\mathbf{C},\mathbf{B}\}=0,\\&(-1)^{rp}\,\{\{\mathbf{B},\mathbf{C}\},\mathbf{D}\}+(-1)^{pq}\,\{\{\mathbf{C},\mathbf{D}\},\mathbf{B}\}+(-1)^{qr}\,\{\{\mathbf{D},\mathbf{B}\},\mathbf{C}\}=0.\end{aligned} \tag{3.8}$$

Here $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are completely skew-symmetric tensor fields of types $(1,p)$, $(1,q)$, and $(1,r)$ respectively (such fields are natural to call *vector valued differential forms*). Due to the relationships (3.8) the set of vector valued differential forms has a structure of graded Lie superalgebra over real numbers.

Theorem 3.2. *Lie derivative* $L_{\boldsymbol{\eta}}$ *along arbitrary vector field* $\boldsymbol{\eta}$ *is a differentiation of degree zero in Lie superalgebra of vector valued differential forms on* M.

The statement of the theorem 3.2 is expressed by the following equality being the Leibniz identity for Froelicher-Nijenhuis bracket:

$$L_{\boldsymbol{\eta}}\{\mathbf{B},\mathbf{C}\}=\{L_{\boldsymbol{\eta}}\mathbf{B},\mathbf{C}\}+\{\mathbf{B},L_{\boldsymbol{\eta}}\mathbf{C}\}. \tag{3.9}$$

For the fields (3.3) the identity (3.9) can be proved by direct calculations based on the formula (3.4) and commutational and anticommutational identities

$$L_{\boldsymbol{\eta}}\circ d-d\circ L_{\boldsymbol{\eta}}=0,\qquad L_{\boldsymbol{\eta}}\circ\iota_{\boldsymbol{\xi}}-\iota_{\boldsymbol{\xi}}\circ L_{\boldsymbol{\eta}}=\iota_{[\boldsymbol{\eta},\,\boldsymbol{\xi}]},\qquad \iota_{\boldsymbol{\xi}}\circ d+d\circ\iota_{\boldsymbol{\xi}}=L_{\boldsymbol{\xi}}$$

from [9]. For the case of arbitrary skew-symmetric tensor fields $\mathbf{B}$ and $\mathbf{C}$ of types $(1,p)$ and $(1,q)$ formula (3.9) is expanded by $\mathbb{R}$-linearity with the use of expansions

$$\mathbf{B}=\sum_{i=1}^{s_1}\mathbf{b}_i\otimes\beta_i,\qquad\qquad \mathbf{C}=\sum_{j=1}^{s_2}\mathbf{c}_j\otimes\beta_j.$$

Denote by $\mathcal{A}=\langle\mathbf{A}^q,\ q\in\mathbb{Z}\,|\,\otimes,C,\{*,*\}\rangle$ the linear span for the closure of the set of integer powers of the operator field $\mathbf{A}$ with respect to the operations of tensor product, contraction, and Froelicher-Nijenhuis bracket[1]. $\mathcal{A}$ is $\mathbb{R}$-subalgebra in the algebra of tensor fields on M. For any two fields $\mathbf{B}$ and $\mathbf{C}$ from $\mathcal{A}$ their Froelicher-Nijenhuis bracket $\{\mathbf{B},\mathbf{C}\}$ (if it is defined) is an element of $\mathcal{A}$. Tensor algebra $\mathcal{A}$ generated by the field $\mathbf{A}$ contains Nijenhuis tensor $\mathbf{N}=\{\mathbf{A},\mathbf{A}\}/2$ and Haantjes tensor from [15] used in [16] for to formulate diagonalizability criterion for the matrix A in (3.2). Moreover this algebra contains semihamiltonity tensor $\mathbf{P}$ (tensor of hydrodynamic integrability) constructed in [12]. From (2.5) and (3.9) we derive the following theorem characterizing tensor algebra $\mathcal{A}$.

Theorem 3.3. *Suppose that $\mathbf{B}$ is an arbitrary tensor field from the tensor algebra $\mathcal{A}=\langle\mathbf{A}^q,\ q\in\mathbb{Z}\,|\,\otimes,C,\{*,*\}\rangle$ generated by operator field $\mathbf{A}$ with components A^i_j given by the system of equations* (1.1). *Then Lie derivative of $\mathbf{B}$ along point symmetry field $\boldsymbol{\eta}$ of the system* (1.1) *is zero, i. e. $L_{\boldsymbol{\eta}}(\mathbf{B})=0$.*

4. Tensor fields generated by affine connection.

Affine connection Γ is a second geometric structure on M given by the system of equations (1.1). It defines tensor field of curvature $\mathbf{R}$ with components (2.8).

Theorem 4.1. *Let $\mathbf{R}$ be the curvature tensor defined by affine connection Γ from* (1.1)*. Then its Lie derivative along the field of point symmetry for the system of equations* (1.1) *is zero, i. e. $L_{\boldsymbol{\eta}}(\mathbf{B})=0$.*

Proof. In order to prove this theorem we shall use well known commutational relationship for covariant derivatives (see [17]):

$$[\nabla_{\mathbf{X}},\,\nabla_{\mathbf{Y}}]\mathbf{Z}=\nabla_{[\mathbf{X},\,\mathbf{Y}]}\mathbf{Z}+\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}. \tag{4.1}$$

Here $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are three arbitrary vector fields on M, while $\mathbf{R}(\mathbf{X},\mathbf{Y})$ is the operator field resulting by contraction of curvature tensor $\mathbf{R}$ and vector fields $\mathbf{X}$ and $\mathbf{Y}$ with respect to last two indices in $\mathbf{R}$. Let's apply Lie derivative $L_{\boldsymbol{\eta}}$ to both sides of the equality (4.1). In left hand side we obtain

$$L_{\boldsymbol{\eta}}[\nabla_{\mathbf{X}},\,\nabla_{\mathbf{Y}}]\mathbf{Z}=L_{\boldsymbol{\eta}}\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}-L_{\boldsymbol{\eta}}\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}. \tag{4.2}$$

Now in the expression $L_{\boldsymbol{\eta}}\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}$ we transpose Lie derivative $L_{\boldsymbol{\eta}}$ first with covariant derivative $\nabla_{\mathbf{X}}$, then with $\nabla_{\mathbf{Y}}$. As a result we have

$$\begin{aligned}L_{\boldsymbol{\eta}}\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}&=[L_{\boldsymbol{\eta}},\,\nabla_{\mathbf{X}}]\nabla_{\mathbf{Y}}\mathbf{Z}+\nabla_{\mathbf{X}}L_{\boldsymbol{\eta}}\nabla_{\mathbf{Y}}\mathbf{Z}=\\&=[L_{\boldsymbol{\eta}},\,\nabla_{\mathbf{X}}]\nabla_{\mathbf{Y}}\mathbf{Z}+\nabla_{\mathbf{X}}[L_{\boldsymbol{\eta}},\,\nabla_{\mathbf{Y}}]\mathbf{Z}+\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}L_{\boldsymbol{\eta}}(\mathbf{Z}).\end{aligned}$$

Taking into account (2.9) we come to the formula

$$L_{\boldsymbol{\eta}}\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}=\nabla_{[\boldsymbol{\eta},\,\mathbf{X}]}\nabla_{\mathbf{Y}}\mathbf{Z}+\nabla_{\mathbf{X}}\nabla_{[\boldsymbol{\eta},\,\mathbf{Y}]}\mathbf{Z}+\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}L_{\boldsymbol{\eta}}(\mathbf{Z}). \tag{4.3}$$

[1]permutation of indices are also assumed to be in the list of operations generating algebra $\mathcal{A}$. This means that $\mathcal{A}$ is closed with respect to symmetrization and alternation of tensor fields.

For $L_{\boldsymbol\eta}\nabla_{\mathbf Y}\nabla_{\mathbf X}\mathbf Z$ there is an analogous formula

$$L_{\boldsymbol\eta}\nabla_{\mathbf Y}\nabla_{\mathbf X}\mathbf Z=\nabla_{[\boldsymbol\eta,\,\mathbf Y]}\nabla_{\mathbf X}\mathbf Z+\nabla_{\mathbf Y}\nabla_{[\boldsymbol\eta,\,\mathbf X]}\mathbf Z+\nabla_{\mathbf Y}\nabla_{\mathbf X}L_{\boldsymbol\eta}(\mathbf Z). \tag{4.4}$$

Let's subtract (4.4) from (4.3) and let's apply the identity (4.1). As a result we can rewrite the equality (4.2) as follows:

$$\begin{aligned}L_{\boldsymbol\eta}[\nabla_{\mathbf X},\,\nabla_{\mathbf Y}]\mathbf Z&=\nabla_{[[\boldsymbol\eta,\,\mathbf X],\,\mathbf Y]}\mathbf Z+\mathbf R([\boldsymbol\eta,\,\mathbf X],\mathbf Y)\mathbf Z+\nabla_{[\mathbf X,\,[\boldsymbol\eta,\,\mathbf Y]]}\mathbf Z\,+\\&+\mathbf R(\mathbf X,[\boldsymbol\eta,\,\mathbf Y])\mathbf Z+\nabla_{[\mathbf X,\,\mathbf Y]}L_{\boldsymbol\eta}(\mathbf Z)+\mathbf R(\mathbf X,\mathbf Y)L_{\boldsymbol\eta}(\mathbf Z).\end{aligned}\tag{4.5}$$

Applying Lie derivative $L_{\boldsymbol\eta}$ to last two summands in right hand side of the identity (4.1) and taking into account the commutational relationship (2.9) together with Jacobi identity for commutator of vector fields we now obtain

$$L_{\boldsymbol\eta}(\nabla_{[\mathbf X,\,\mathbf Y]}\mathbf Z)=\nabla_{[[\boldsymbol\eta,\,\mathbf X],\,\mathbf Y]}\mathbf Z+\nabla_{[\mathbf X,\,[\boldsymbol\eta,\,\mathbf Y]]}\mathbf Z+\nabla_{[\mathbf X,\,\mathbf Y]}L_{\boldsymbol\eta}(\mathbf Z). \tag{4.6}$$

$$\begin{aligned}L_{\boldsymbol\eta}(\mathbf R(\mathbf X,\mathbf Y)\mathbf Z)&=L_{\boldsymbol\eta}(\mathbf R)(\mathbf X,\mathbf Y)\mathbf Z+\mathbf R([\boldsymbol\eta,\,\mathbf X],\mathbf Y)\mathbf Z\,+\\&+\,\mathbf R(\mathbf X,[\boldsymbol\eta,\,\mathbf Y])\mathbf Z+\mathbf R(\mathbf X,\mathbf Y)L_{\boldsymbol\eta}(\mathbf Z).\end{aligned}\tag{4.7}$$

When subtracting (4.6) and (4.7) from (4.5) all summands in right hand side are canceled, except for only one. Here is the ultimate result of the above calculations:

$$L_{\boldsymbol\eta}(\mathbf R)(\mathbf X,\mathbf Y)\mathbf Z=0.$$

Since $\mathbf X$, $\mathbf Y$, and $\mathbf Z$ are arbitrary vector fields, we obtain the equality $L_{\boldsymbol\eta}(\mathbf R)=0$, which was required to prove the theorem. $\square$

Covariant derivative $\nabla_{\mathbf Y}$ is closely connected with the concept of *covariant differential*. For the tensor field $\mathbf W$ of type (r,s) its covariant differential is a tensor field $\nabla\mathbf W$ of type $(r,s+1)$ such that when being contracted with vector field $\mathbf Y$ yields the field of covariant derivative $\nabla_{\mathbf Y}\mathbf W$:

$$\nabla_{\mathbf Y}\mathbf W=C(\mathbf Y\otimes\nabla\mathbf W). \tag{4.8}$$

Let's apply Lie derivative $L_{\boldsymbol\eta}$ along the field of point symmetry of (1.1) to both sides of the equality (4.8). And take into account (2.9) doing this:

$$\begin{aligned}L_{\boldsymbol\eta}(\nabla_{\mathbf Y}\mathbf W)=&\nabla_{[\boldsymbol\eta,\,\mathbf Y]}\mathbf W+\nabla_{\mathbf Y}L_{\boldsymbol\eta}(\mathbf W)=\\&=C([\boldsymbol\eta,\,\mathbf Y]\otimes\nabla\mathbf W)+C(\mathbf Y\otimes\nabla L_{\boldsymbol\eta}(\mathbf W)).\end{aligned}$$

$$L_{\boldsymbol\eta}C(\mathbf Y\otimes\nabla\mathbf W)=C([\boldsymbol\eta,\,\mathbf Y]\otimes\nabla\mathbf W)+C(\mathbf Y\otimes L_{\boldsymbol\eta}(\nabla\mathbf W)).$$

Equating right hand sides of two resulting relationships we obtain

$$C(\mathbf Y\otimes\nabla L_{\boldsymbol\eta}(\mathbf W))=C(\mathbf Y\otimes L_{\boldsymbol\eta}(\nabla\mathbf W)).$$

Now remember that $\mathbf{Y}$ is an arbitrary vector field. This leads us to the equality $\nabla L_{\boldsymbol{\eta}}(\mathbf{W}) = L_{\boldsymbol{\eta}}(\nabla\mathbf{W})$, which can be written as

$$[L_{\boldsymbol{\eta}},\, \nabla] = 0 \tag{4.9}$$

since $\mathbf{W}$ is also an arbitrary tensor field.

Theorem 4.2. *Lie derivative $L_{\boldsymbol{\eta}}$ along the field of point symmetry for the system of equations* (1.1) *is commutating with covariant differential ∇ defined by the affine connection with components Γ^j_{rs} from* (1.1).

Let's add the curvature tensor to the set of integer powers of the operator field $\mathbf{A}$ and let's consider linear span for the closure of the resulting set with respect to the operations of tensor product, contraction, Froelicher-Nijenhuis bracket, and covariant differential[1]:

$$\mathcal{R} = \langle \mathbf{R},\, \mathbf{A}^q,\ q \in \mathbb{Z} \,|\, \otimes, C, \{*,*\}, \nabla\rangle. \tag{4.10}$$

Theorem 4.3. *If tensor algebra $\mathcal{R}$ in* (4.10) *is generated by the operator field $\mathbf{A}$ and affine connection Γ with components taken from* (1.1)*, then for any tensor field $\mathbf{B}$ from $\mathcal{R}$ its Lie derivative $L_{\boldsymbol{\eta}}(\mathbf{B})$ along the field of point symmetry $\boldsymbol{\eta}$ for this system of equations is zero, i. e. $L_{\boldsymbol{\eta}}(\mathbf{B}) = 0$.*

Tensor algebra $\mathcal{R}$ contains two tensor fields important for the further study of the equations (1.1). These are the fields $\mathbf{R}$ and $\mathbf{S}$ with components

$$R_{ij} = \sum_{k=1}^{n} R^k_{ikj}, \qquad\qquad S_{ij} = \sum_{k=1}^{n} R^k_{kij}. \tag{4.11}$$

First of them is a field of Ricci tensor. For arbitrary affine connection it has symmetric part and antisymmetric part as well:

$$\hat{R}_{ij} = \frac{R_{ij} + R_{ji}}{2}, \qquad\qquad \tilde{R}_{ij} = \frac{R_{ij} - R_{ji}}{2}. \tag{4.12}$$

Tensor fields $\hat{\mathbf{R}}$ and $\tilde{\mathbf{R}}$ with components (4.12) belong to the tensor algebra $\mathcal{R}$. For $\hat{\mathbf{R}}$ and $\tilde{\mathbf{R}}$ we have the relation $\mathbf{S} = 2\,\tilde{\mathbf{R}}$, which follows from well-known identity $R^s_{ijk} + R^s_{jki} + R^s_{kij} = 0$. The latter is the consequence of symmetry $\Gamma^j_{rs} = \Gamma^j_{sr}$.

5. Symmetries with stationary point.

Let $\boldsymbol{\eta}$ be a field of point symmetry for the system of equations (1.1). It gives rise to the local one-parametric group of transformations $\varphi_\tau : M \to M$. In local coordinates $y^1, \ldots, y^n$ on M these transformations are represented by functions

$$\begin{aligned} &x^1 = x^1(\tau, y^1, \ldots, y^n),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &x^n = x^n(\tau, y^1, \ldots, y^n). \end{aligned} \tag{5.1}$$

[1] permutation of indices here are assumed to be in the set of generating operations for the algebra $\mathcal{R}$ as well as in case of algebra $\mathcal{A}$.

Here arguments $y^1, \ldots, y^n$ are the coordinates of initial point $p \in M$, while the values of functions $x^1, \ldots, x^n$ are the coordinates of transformed point $\varphi_\tau(p)$. Functions (5.1) are the solutions for the system of ordinary differential equations

$$\begin{gathered} (x^1)'_\tau = \eta^1(x^1, \ldots, x^n),\\ \dots\dots\dots\dots\dots\dots\\ (x^n)'_\tau = \eta^n(x^1, \ldots, x^n) \end{gathered} \tag{5.2}$$

such that they solve Cauchy problem with the following initial data:

$$x^1(\tau)\Bigr|_{\tau=0} = y^1, \ \ldots, \ x^n(\tau)\Bigr|_{\tau=0} = y^n. \tag{5.3}$$

Initial data (5.3) mean that $\varphi_\tau \colon M \to M$ is identical map for $\tau = 0$.

Let p_0 be a stationary point for the maps φ_τ, i. e. $\varphi_\tau(p_0) = p_0$ for all τ. If $y^1_0, \ldots, y^n_0$ are coordinates of such point, then functions

$$x^1(\tau) = y^1_0 = \text{const}, \ \ldots, \ x^n(\tau) = y^n_0 = \text{const}$$

should satisfy the system of equations (5.2) and initial conditions (5.3). Substitution of these functions into (5.2) leads us to the following well-known fact.

Lemma 5.1. *Point p on M is a stationary point of local one-parametric group of transformations $\varphi_\tau \colon M \to M$ if and only if corresponding vector field $\boldsymbol\eta$ vanishes at this point.*

If p_0 is a vanishing point for the vector field $\boldsymbol\eta$, then we can consider the differential of the map φ_τ. Due to $\varphi_\tau(p_0) = p_0$ it is linear operator in tangent space $T_{p_0}(M)$ to the manifold M at the point p_0. In local coordinates $y^1, \ldots, y^n$ operator $\varphi_{\tau *} \colon T_{p_0}(M) \to T_{p_0}(M)$ is expressed by the Jacobi matrix for the system of functions (5.1) at stationary point p_0:

$$\Phi(\tau) = \begin{Vmatrix} \dfrac{\partial x^1}{\partial y^1} & \ldots & \dfrac{\partial x^1}{\partial y^n}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial x^n}{\partial y^1} & \ldots & \dfrac{\partial x^n}{\partial y^n} \end{Vmatrix} \tag{5.4}$$

Components of matrix (5.4) are smooth functions of parameter τ. Denote by F the matrix composed by derivatives in τ of the components of matrix $\Phi(\tau)$ for $\tau = 0$. In order to find the matrix $\Phi(\tau)$ we should solve Cauchy problem (5.3) for the equations (5.2). However, components of the matrix F can be expressed through components of vector field $\boldsymbol\eta$ without solving the equations (5.2):

$$F^i_j = \frac{\partial \Phi^i_j}{\partial \tau}\biggr|_{\tau=0} = \frac{\partial \eta^i}{\partial y^j}\biggr|_{p=p_0}. \tag{5.5}$$

Matrices $\Phi(\tau)$ with various values of τ form one-parametric matrix group because $\Phi(\tau_1+\tau_2)=\Phi(\tau_1)\,\Phi(\tau_2)$. These matrices are analogs of rotation matrices, while matrix F with components (5.5) is an analog of infinitesimal rotation around the point p_0. From $\Phi(\tau_1+\tau_2)=\Phi(\tau_1)\,\Phi(\tau_2)$ we derive differential equation for $\Phi(\tau)$:

$$\Phi'_\tau = F\,\Phi. \tag{5.6}$$

The solution of (5.6) is a matrix exponent $\Phi(\tau)=\exp(F\,\tau)$.

We can give another (more direct) interpretation for the matrix F. Let's consider Lie derivative $L_{\boldsymbol\eta}$. Being a differentiation in the set of vector fields Lie derivative $L_{\boldsymbol\eta}$ satisfies the following relationships:

$$L_{\boldsymbol\eta}(\mathbf{X}+\mathbf{Y}) = L_{\boldsymbol\eta}(\mathbf{X}) + L_{\boldsymbol\eta}(\mathbf{Y}), \qquad L_{\boldsymbol\eta}(\psi\cdot\mathbf{X}) = \psi\cdot L_{\boldsymbol\eta}(\mathbf{X}) + (\boldsymbol\eta\psi)\cdot\mathbf{X}.$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are vector fields, while ψ is a scalar field. When written at the point p_0, due to $\boldsymbol\eta(p_0)=0$, the above relationships look like $L_{\boldsymbol\eta}(\mathbf{X}+\mathbf{Y}) = L_{\boldsymbol\eta}(\mathbf{X})+L_{\boldsymbol\eta}(\mathbf{Y})$ and $L_{\boldsymbol\eta}(\psi\cdot\mathbf{X}) = \psi\cdot L_{\boldsymbol\eta}(\mathbf{X})$. This means that $L_{\boldsymbol\eta}$ acts as a linear operator

$$L_{\boldsymbol\eta}\colon T_{p_0}(M)\to T_{p_0}(M) \tag{5.7}$$

in the tangent space to M at the point p_0. By direct calculations in local coordinates one can easily check that the matrix of the operator (5.7) has the components defined by the right hand side of (5.5).

6. Estimate for dimension of symmetry algebra.

Vector fields of point symmetries for the system of equations (1.1) constitute Lie algebra over real numbers (see for instance [7] or [8]). In this paper we consider a subalgebra that consists of special point symmetries (2.1), denote it by $\mathcal{G}$. Elements of $\mathcal{G}$ are interpreted as vector fields on the manifold M. Let $\boldsymbol\eta\in\mathcal{G}$. Consider the equation (4.9), which is equivalent to (2.9). In local coordinates $y^1,\,\ldots\,,\,y^n$ it is written in form of the system of equations (2.6) with respect to the components of vector fields $\boldsymbol\eta$. Let's rewrite this system of equations as

$$\frac{\partial F^i_s}{\partial y^r} = \sum_{k=1}^{n}\Gamma^k_{rs}\,F^i_k - \sum_{k=1}^{n}\left(\frac{\partial\Gamma^i_{rs}}{\partial y^k}\,\eta^k + \Gamma^i_{ks}\,F^k_r + \Gamma^i_{rk}\,F^k_s\right) \tag{6.1}$$

and let's complete it with the equations defining F^i_r:

$$\frac{\partial\eta^i}{\partial y^r} = F^i_r. \tag{6.2}$$

Equations (6.1) and (6.2) in the aggregate form complete system of Pfaff equations with respect to n^2+n functions $F^i_s(y^1,\ldots,y^n)$ and $\eta^i(y^1,\ldots,y^n)$. It's known that each solution of such system is uniquely defined by the initial data at some point:

$$F^i_s\Big|_{p=p_0} = F^i_s(0), \qquad\qquad \eta^i\Big|_{p=p_0} = \eta^i(0). \tag{6.3}$$

However, not for each initial data (6.3) one can find appropriate solution of the equations (6.1) and (6.2). Some constraints there appear since system o Pfaff equations my be not completely compatible. Therefore we have the theorem.

Theorem 6.1. *For any system of equations* (1.1) *the Lie algebra of special point symmetries* (2.1) *is finite dimensional and its dimension is not greater than* $n(n+1)$.

By proving the above theorem 6.1 we reproduce partially (on a level of Lie algebras) the following well-known result (see [10], chapter 4, § 1).

Theorem 6.2. *Let M be n-dimensional manifold with affine connection. Then group of affine transformations of the manifold M is a Lie group of the dimension not greater than $n(n+1)$.*

7. Case of maximal degeneration.

The estimate in theorem 6.1 is exact. For to prove this we shall find the condition for complete compatibility of the system of Pfaff equations (6.1) and (6.2). For the sake of brevity let's introduce the notations

$$\Omega^{ij}_{rsq} = \Gamma^{j}_{rs}\,\delta^{i}_{q} - \Gamma^{i}_{qs}\,\delta^{j}_{r} - \Gamma^{i}_{qr}\,\delta^{j}_{s}, \tag{7.1}$$

where δ^i_j is a Kronecker's delta-symbol. Then we can write (6.1) and (6.2) as

$$\frac{\partial F^i_s}{\partial y^r} = \sum_{j=1}^{n}\sum_{q=1}^{n}\Omega^{ij}_{rsq}\,F^q_j - \sum_{k=1}^{n}\frac{\partial\Gamma^i_{rs}}{\partial y^k}\,\eta^k, \qquad\qquad \frac{\partial\eta^i}{\partial y^r} = F^i_r. \tag{7.2}$$

Compatibility condition for the equations (7.2) is obtained by equating partial derivatives calculated by virtue of these equations:

$$\frac{\partial^2 F^i_s}{\partial y^p\,\partial y^r} = \frac{\partial^2 F^i_s}{\partial y^r\,\partial y^p}, \qquad\qquad \frac{\partial^2\eta^i}{\partial y^p\,\partial y^r} = \frac{\partial^2\eta^i}{\partial y^r\,\partial y^p}. \tag{7.3}$$

Note that second part of equalities (7.3) are identically fulfilled due to the symmetry $\Gamma^i_{pr} = \Gamma^i_{rp}$ and $\Omega^{ij}_{rpq} = \Omega^{ij}_{prq}$. For to write complete compatibility condition we should equate coefficients of each F^q_j and each η^k in the first part of (7.3) assuming F^q_j and η^k to be independent variables. This yields

$$\begin{aligned}\frac{\partial\Omega^{ij}_{rsq}}{\partial y^p}&+\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Omega^{i\beta}_{rs\alpha}\,\Omega^{\alpha j}_{p\beta q} - \frac{\partial\Gamma^i_{rs}}{\partial y^q}\,\delta^j_p = \\ &= \frac{\partial\Omega^{ij}_{psq}}{\partial y^r} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\Omega^{i\beta}_{ps\alpha}\,\Omega^{\alpha j}_{r\beta q} - \frac{\partial\Gamma^i_{ps}}{\partial y^q}\,\delta^j_r,\end{aligned} \tag{7.4}$$

$$\frac{\partial^2\Gamma^i_{rs}}{\partial y^p\,\partial y^k} + \sum_{q=1}^{n}\sum_{j=1}^{n}\Omega^{ij}_{rsq}\,\frac{\partial\Gamma^q_{pj}}{\partial y^k} = \frac{\partial^2\Gamma^i_{ps}}{\partial y^r\,\partial y^k} + \sum_{q=1}^{n}\sum_{j=1}^{n}\Omega^{ij}_{psq}\,\frac{\partial\Gamma^q_{rj}}{\partial y^k}. \tag{7.5}$$

Let's substitute (7.1) into (7.4) and do contract indices j and q. As a result of this operation we obtain the equality

$$\frac{\partial \Gamma^i_{rs}}{\partial y^p}+\sum_{\beta=1}^{n}\Gamma^{\beta}_{rs}\,\Gamma^i_{p\beta}=\frac{\partial \Gamma^i_{ps}}{\partial y^r}+\sum_{\beta=1}^{n}\Gamma^{\beta}_{ps}\,\Gamma^i_{r\beta},\tag{7.6}$$

which means $\mathbf{R}=0$ (compare with (2.8)). Upon substituting (7.1) into (7.5) this relationship appears to be differential consequence of (7.6): it can be derived from (7.6) by differentiating both sides with respect to y^k. As for the relationship (7.4) itself, after substituting (7.1) into it and after collecting similar term it takes the form of linear combination of four equalities, each obtained by some permutation of indices in (7.6). The result of these computations we formulate as a lemma.

Lemma 7.1. *System of Pfaff equations* (6.1) *and* (6.2) *is completely compatible if and only if affine connection with components* Γ^i_{rs} *is a flat connection with zero curvature tensor.*

The following fact is well-known in geometry (see [17]): if affine connection is flat, then there exist a local coordinate system $y^1,\,\ldots\,,\,y^n$ on M such that all connection components are identically zero in these coordinates (they are called *euclidean coordinates*). We shall use such euclidean coordinates in order to solve Pfaff equations (6.1) and (6.2). Here these equations become very simple:

$$\frac{\partial F^i_s}{\partial y^r}=0,\qquad\qquad \frac{\partial \eta^i}{\partial y^r}=F^i_r.\tag{7.7}$$

Cauchy problem (6.3) for the equations (7.7) is solvable for any initial data $F^i_s(0)$ and $\eta^i(0)$. If we suppose the coordinates of p_0 to be zero, then the solution of this Cauchy problem is given by the following linear functions:

$$F^i_s=F^i_s(0)=\text{const},\qquad\qquad \eta^i=\eta^i(0)+\sum_{s=1}^{n}F^i_s(0)\cdot y^s.\tag{7.8}$$

Taking components of the field $\boldsymbol{\eta}$ from (7.8) we substitute this field into the equation (2.5). When written in local coordinates, this equation has the form (2.2). As a result we obtain the equation

$$\sum_{k=1}^{n}\left(A^i_k\,F^k_j(0)-F^i_k(0)\,A^k_j\right)+\sum_{k=1}^{n}\left(\sum_{s=1}^{n}F^k_s(0)\,y^s+\eta^k(0)\right)\frac{\partial A^i_j}{\partial y^k}=0.\tag{7.9}$$

For the arbitrary vector field $\boldsymbol{\eta}$ with components (7.8) to be the field of point symmetry for (1.1) the equations (7.9) should be fulfilled identically for arbitrary values of constants $F^i_s(0)$ and $\eta^i(0)$. Let's substitute $F^i_s(0)=0$ into (7.9) and take into account that $\eta^i(0)$ are arbitrary constants. This yields $A^i_j=\text{const}$ and

$$\sum_{k=1}^{n}A^i_k\,F^k_j(0)=\sum_{k=1}^{n}F^i_k(0)\,A^k_j.\tag{7.10}$$

The relationships (7.10) mean that matrix A should commutate with arbitrary matrix F whose components are $F^i_j(0)$. Therefore A can differ from unit matrix only by scalar factor.

Theorem 7.1. *System of equation* (1.1) *with nondegenerate matrix A has an algebra of point symmetries* (2.1) *of maximal dimension* $n(n+1)$ *if and only if A is a scalar matrix ($A^i_j = a\,\delta^i_j$) with constant factor* $a = \text{const}$ *and if affine connection Γ with components Γ^j_{rs} is flat (i. e. its curvature is zero).*

This result is in agreement with the following well-known geometric fact (see [10], chapter 4, § 1).

Theorem 7.2. *Let M be n-dimensional manifold with affine connection. The dimension of the group of affine transformations is equal to* $n(n+1)$ *exactly if and only if M is an ordinary flat affine space $\mathbb{A}_n$ with natural flat affine connection.*

Systems of equations (1.1) described by the theorem 7.1 constitute maximally symmetric, and therefore maximally simple (maximally degenerate), subclass of such systems. For any of them we can find point transformation (1.4) breaking this system into separate equations of the form

$$\frac{\partial y^i}{\partial t} = a\,\frac{\partial^2 y^i}{\partial x^2}, \quad i = 1, \ldots, n, \tag{7.11}$$

where a is a common constant. Geometry associated with the equations (7.11) is also maximally simple. In order to find the equations with more interesting geometry further we consider the cases when algebra of point symmetries of such equations is not maximal, but is extremal in some sense as described below.

8. Cases of intermediate degeneration.

Let p_0 be some fixed point on the manifold M. Vector fields from $\mathcal{G}$ vanishing at the point p_0 constitute subalgebra in $\mathcal{G}$. Denote it by $\mathcal{G}(p_0)$. For the dimension of factorspace $\mathcal{G}/\mathcal{G}(p_0)$ (not factoralgebra) we have an obvious estimate

$$\dim(\mathcal{G}/\mathcal{G}(p_0)) \leqslant \dim M = n. \tag{8.1}$$

Estimate (8.1) can be obtained from the statement of Cauchy problem (6.3) for Pfaff equations (6.1) and (6.2).

When Pfaff equations (6.1) and (6.2) are not completely compatible the corresponding Cauchy problem for them can be solved not for all initial data in (6.3). Restrictions are due to differential consequences of (6.1) and (6.2). One of such differential consequences can be formulated in terms of symmetric part of Ricci tensor: $L_{\boldsymbol{\eta}}(\hat{\mathbf{R}}) = 0$. Let's write this equation in local coordinates for the field of point symmetry $\boldsymbol{\eta} \in \mathcal{G}(p_0)$:

$$\sum_{k=1}^{n} \hat{R}_{ik}\,F^k_j(0) + \sum_{k=1}^{n} \hat{R}_{kj}\,F^k_i(0) = 0. \tag{8.2}$$

Here $F^i_j(0)$ are the components of the matrix defined in (5.5). These quantities are initial data in Cauchy problem (6.3).

Ricci tensor $\hat{\mathbf{R}}$ is symmetric tensor field of type $(0,2)$, i. e. it is a field of quadratic forms. Denote by $\hat{R}(\mathbf{X},\mathbf{Y})$ the symmetric bilinear form given by the field $\hat{\mathbf{R}}$. Let m be the sum of positive and negative indices in signature of this form (see for instance [18]). In other words m is the rank of matrix formed by the components of Ricci tensor $\hat{\mathbf{R}}$.

Definition 8.1. Say that the system of equations (1.1) belongs to m-th case of intermediate degeneration ($1 \leqslant m \leqslant n-1$) if the rank of symmetric part of Ricci tensor $\hat{\mathbf{R}}$ defined by affine connection Γ with components Γ^j_{rs} is equal to m everywhere on the manifold M.

Case of maximal degeneration corresponds to $m = 0$, while in case of general position we have $m = n$. In order to find an estimate for the dimension of algebra of point symmetries depending on m we write (8.2) in invariant form

$$\hat{R}(\mathbf{F}\mathbf{X},\mathbf{Y}) = -\hat{R}(\mathbf{X},\mathbf{F}\mathbf{Y}), \tag{8.3}$$

which do not refer to local coordinates. This equality should be fulfilled for two arbitrary vectors $\mathbf{X}$ and $\mathbf{Y}$ from tangent space $V = T_{p_0}(M)$. Here $\mathbf{F} = L_{\boldsymbol{\eta}}$ is a linear operator from (5.7). Denote by W kernel of quadratic form $\hat{R}$ (see [18]):

$$W = \{\mathbf{X} \in V : \ \ \hat{R}(\mathbf{X},\mathbf{Y}) = 0 \ \text{ for all } \ \mathbf{Y} \in V\}. \tag{8.4}$$

Its easy to calculate the dimension of the subspace (8.4): $\dim W = n - m$.

From (8.3) we find that W is invariant under the action of the operator $\mathbf{F}$, i. e. $\mathbf{X} \in W$ implies $\mathbf{F}\mathbf{X} \in W$. Operators preserving W constitute a subspace of the dimension $n^2 - m(n-m)$ in the space $\operatorname{End}(V)$ of all linear operators in V. Denote it by $\operatorname{End}(V|W)$. For the operators from $\operatorname{End}(V|W)$ the concept of *restriction to* W and the concept of *factoroperator* are defined:

$$\mathbf{F}\Big|_W : W \to W, \qquad\qquad \hat{\mathbf{F}} = \mathbf{F}\Big|_{V/W} : V/W \to V/W. \tag{8.5}$$

The action of latter one to cosets relative to W is given by the formula

$$\hat{\mathbf{F}}\operatorname{Cl}_W(\mathbf{X}) = \operatorname{Cl}_W(\mathbf{F}\mathbf{X})$$

(see more details in [18]). Replacing operator $\mathbf{F} \in \operatorname{End}(V|W)$ by factoroperator $\hat{\mathbf{F}} \in \operatorname{End}(V/W)$ we factorize over the operators mapping V into W:

$$\operatorname{End}(V/W) \cong \operatorname{End}(V|W)/\operatorname{Hom}(V,W). \tag{8.6}$$

Subspace W is a kernel of $\hat{R}$. This implies that $\hat{R}$ induces nondegenerate symmetric bilinear form $\hat{R}$ in factorspace V/W. It is defined by formula

$$\hat{R}(\hat{\mathbf{X}},\hat{\mathbf{Y}}) = \hat{R}(\mathbf{X},\mathbf{Y}), \tag{8.7}$$

where $\hat{\mathbf{X}} = \mathrm{Cl}_W(\mathbf{X})$ and $\hat{\mathbf{Y}} = \mathrm{Cl}_W(\mathbf{Y})$ are cosets relative to the subspace W. In whole, the equality (8.3) for the operators $\mathbf{F} \in \mathrm{End}(V)$ appears to be equivalent to the following two conditions:

(1) operator $\mathbf{F}$ belongs to the subspace $\mathrm{End}(V|W)$ that consists of operators having $W = \mathrm{Ker}\,\hat{R}$ as invariant subspace;
(2) factoroperator $\hat{F}$ in (8.5) is skew-symmetric with respect to the symmetric bilinear form (8.7), i. e. the equality $\hat{R}(\hat{\mathbf{F}}\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = -\hat{R}(\hat{\mathbf{X}}, \hat{\mathbf{F}}\hat{\mathbf{Y}})$ holds for two arbitrary vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ from factorspace V/W.

The dimension of subspace formed by the operators skew-symmetric with respect to nondegenerate bilinear form (8.7) is given by the formula

$$\dim \hat{\mathcal{F}}_{\text{skew}} = \frac{m(m-1)}{2}.$$

We know the dimension of the space $\mathrm{Hom}(V, W)$, over which we factorize in (8.6). Thereby we can calculate the dimension of the space of operators satisfying (8.3):

$$\dim \mathcal{F} = \dim \mathrm{Hom}(V, W) + \dim \hat{\mathcal{F}}_{\text{skew}} = n(n-m) + \frac{m(m-1)}{2}. \tag{8.8}$$

Let's combine (8.8) with (8.1). This gives an estimate for dimensions of symmetry algebras of the system of equations (1.1) in all cases of intermediate degeneration:

$$\dim(\mathcal{G}) \leqslant n(n+1-m) + \frac{m(m-1)}{2}. \tag{8.9}$$

This estimate is in agreement with the following theorem from [19].

Theorem 8.1. *Maximally movable spaces of affine connection with the symmetric part of Ricci tensor of the rank m possess transitive groups of automorphisms with $n(n+1-m) + m(m-1)/2$ parameters.*

In paper [20] was shown that the estimate (8.9) is exact, some examples of spaces, where the upper bound is reached, were given. Here we give complete description of such spaces and describe the appropriate equations (1.1) for them in each case of intermediate degeneration. For $m = 1$ this was done in [21].

9. Structure of the tensor field S.

Let (1.1) be the system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension

$$\dim(\mathcal{G}) = n(n+1-m) + \frac{m(m-1)}{2}. \tag{9.1}$$

Let $\mathbf{S} = 2\,\tilde{\mathbf{R}}$ be a tensor field of type $(0,2)$ with components (4.11). For this field we state the following lemma.

Lemma 9.1. *Kernel of skew-symmetric bilinear form defined by* $\mathbf{S}$ *contains the kernel of the form defined by symmetric part of Ricci tensor.*

Proof. Tensor field $\mathbf{S}$ belongs to the algebra $\mathcal{R}$. Therefore due to theorem 4.3 we have the relationship $L_{\boldsymbol{\eta}}\mathbf{S} = 0$ satisfied for any field of point symmetry $\boldsymbol{\eta}$ from $\mathcal{G}$. Repeating arguments from section 8 we get the relationship

$$S(\mathbf{F}\mathbf{X}, \mathbf{Y}) = -S(\mathbf{X}, \mathbf{F}\mathbf{Y}) \tag{9.2}$$

analogous to (8.3). The equality (9.1) here means that (9.2) should be fulfilled for all operators satisfying (8.3). Let's take an operator $\mathbf{F} \in \operatorname{Hom}(V, W)$ of special form $\mathbf{F} = \mathbf{w} \otimes \alpha$, where $\mathbf{w}$ is a vector from the kernel of the form $\hat{R}$ and α is some arbitrary linear functional in the space V. Operator $\mathbf{F}$ satisfies the equation (8.3). Therefore it should satisfy the equation (9.2) too. Substituting $\mathbf{F} = \mathbf{w} \otimes \alpha$ into (9.2) we get the following equality:

$$\alpha(\mathbf{X})\, S(\mathbf{w}, \mathbf{Y}) = \alpha(\mathbf{Y})\, S(\mathbf{X}, \mathbf{w}). \tag{9.3}$$

In the space V of dimension $n \geqslant 2$ for any vector $\mathbf{X} \neq 0$ one can find the vector $\mathbf{Y}$ noncollinear to $\mathbf{X}$. For these two vectors there exists a linear functional such that

$$\alpha(\mathbf{X}) = 0, \qquad\qquad \alpha(\mathbf{Y}) = 1. \tag{9.4}$$

Substituting (9.4) into (9.3) we obtain $S(\mathbf{X}, \mathbf{w}) = 0$ for any vector $\mathbf{X} \in V$ and for any vector $\mathbf{w} \in W = \operatorname{Ker} \hat{R}$. This completes the proof of lemma 9.1. $\square$

The condition $n \geqslant 2$ used in proof of lemma 9.1 follows from the inequalities $1 \leqslant m \leqslant n-1$ determining m-th case of intermediate degeneration. If $n = 2$, then skew-symmetric bilinear form with nonzero kernel is identically zero: $\mathbf{S} = 2\,\tilde{\mathbf{R}} = 0$.

Lemma 9.2. *Let* (1.1) *be a system of equations belonging to* m*-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions if* $m = 1$ *or* $m \geqslant 3$*, then* $\mathbf{S} = 2\,\tilde{\mathbf{R}} = 0$.

Proof. Let's use the result of previous lemma 9.1. Due to the inclusion $W \subset \operatorname{Ker} S$ tensor field $\mathbf{S}$ induces a skew-symmetric bilinear form $\tilde{S}$ on the factorspace V/W. It is defined by means of the relationship

$$\tilde{S}(\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = S(\mathbf{X}, \mathbf{Y}), \tag{9.5}$$

where $\hat{\mathbf{X}} = \operatorname{Cl}_W(\mathbf{X})$ and $\hat{\mathbf{Y}} = \operatorname{Cl}_W(\mathbf{Y})$. For $m = 1$ we have $\dim(V/W) = 1$ and remember that skew-symmetric bilinear form in one-dimensional space is identically zero. Therefore $\tilde{S} = 0$. Due to (9.5) this implies $\mathbf{S} = 2\,\tilde{\mathbf{R}} = 0$.

Now let m be greater than 1. The equality (9.1) for the dimension of the algebra of point symmetries $\mathcal{G}$ means that the relationship

$$\tilde{S}(\hat{\mathbf{F}}\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = -\tilde{S}(\hat{\mathbf{X}}, \hat{\mathbf{F}}\hat{\mathbf{Y}}) \tag{9.6}$$

should be fulfilled for arbitrary operator $\hat{\mathbf{F}} \in \operatorname{End}(V/W)$ satisfying the equality $\hat{R}(\hat{\mathbf{F}}\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = -\hat{R}(\hat{\mathbf{X}}, \hat{\mathbf{F}}\hat{\mathbf{Y}})$. By applying the complexification procedure to the factorspace V/W, if necessary, the matrix of nondegenerate symmetric bilinear form $\hat{R}$ from (8.7) can be brought to the unit matrix at the expense of proper choice of base $\mathbf{e}_1, \ldots, \mathbf{e}_m$ in factorspace V/W:

$$\hat{R}(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases} \tag{9.7}$$

When $m \geqslant 3$ we define an operator $\hat{\mathbf{F}}$ by prescribing its action upon the base vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ of the above base in factorspace V/W:

$$\hat{\mathbf{F}}(\mathbf{e}_i) = \begin{cases} \mathbf{e}_q & \text{for} \quad i = k, \\ -\mathbf{e}_k & \text{for} \quad i = q, \\ 0 & \text{for} \quad i \neq k \quad \text{and} \quad i \neq q. \end{cases} \tag{9.8}$$

From (9.7) and (9.8) for operator $\hat{\mathbf{F}}$ we derive $\hat{R}(\hat{\mathbf{F}}\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = -\hat{R}(\hat{\mathbf{X}}, \hat{\mathbf{F}}\hat{\mathbf{Y}})$. Hence (9.6) should be fulfilled for $\hat{\mathbf{F}}$. Taking into account (9.8) from (9.6) we obtain

$$\tilde{S}_{ik} = \tilde{S}(\mathbf{e}_i, \mathbf{e}_k) = -\tilde{S}(\mathbf{e}_i, \hat{\mathbf{F}}\mathbf{e}_q) = \tilde{S}(\hat{\mathbf{F}}\mathbf{e}_i, \mathbf{e}_q) = S(0, \mathbf{e}_q) = 0.$$

Since i, k, q are three arbitrary indices, the above equality yields $\tilde{S}_{ik} = 0$ for all nondiagonal elements in the matrix of bilinear form $\tilde{S}$. Diagonal elements are zero due to skew symmetry. Hence $\tilde{S} = 0$ and $\mathbf{S} = 2\,\tilde{\mathbf{R}} = 0$. Lemma is proved. □

10. Structure of curvature tensor for $n \geqslant 4$.

According to the theorem 4.1 Lie derivative of curvature tensor $\mathbf{R}$ along any field of point symmetry of the system of equations (1.1) is zero. In local coordinates the equation $L_{\boldsymbol{\eta}}(\mathbf{R}) = 0$ is written in the following form:

$$\sum_{k=1}^{n}\left(\eta^k\,\frac{\partial R^i_{jrs}}{\partial y^k} - R^k_{jrs}\,\frac{\partial \eta^i}{\partial y^k} + R^i_{krs}\,\frac{\partial \eta^k}{\partial y^j} + R^i_{jks}\,\frac{\partial \eta^k}{\partial y^r} + R^i_{jrk}\,\frac{\partial \eta^k}{\partial y^s}\right) = 0. \tag{10.1}$$

Let's take and fix some point p_0 on M. For the field of point symmetry $\boldsymbol{\eta}$ from $\mathcal{G}(p_0)$ we can transform (10.1) to the form analogous to (8.3):

$$\mathbf{R}(\mathbf{F}\mathbf{X}, \mathbf{Y})\mathbf{Z} + \mathbf{R}(\mathbf{X}, \mathbf{F}\mathbf{Y})\mathbf{Z} + \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{F}\mathbf{Z} = \mathbf{F}\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}. \tag{10.2}$$

Here $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are three arbitrary vectors from tangent space $V = T_{p_0}(M)$. Let (1.1) be a system of equations belonging to m-th case of intermediate degeneration. Then (9.1) means that the relationship (10.2) is fulfilled for all those operators $\mathbf{F}$, for which (8.3) holds.

In m-th case of intermediate degeneration the kernel of bilinear form $\hat{R}$ in (8.3) is nonzero since $1 \leqslant m \leqslant n-1$. Choosing some nonzero vector $\mathbf{w} \in W = \operatorname{Ker}\hat{R}$ we construct an operator $\mathbf{F} = \mathbf{w} \otimes \alpha$, where α is an arbitrary linear functional in V. Such operator satisfies the equality (8.3). By substituting it into (10.2) we get

$$\begin{aligned} &\alpha(\mathbf{X}) \cdot \mathbf{R}(\mathbf{w}, \mathbf{Y})\mathbf{Z} + \alpha(\mathbf{Y}) \cdot \mathbf{R}(\mathbf{X}, \mathbf{w})\mathbf{Z} + \\ &\quad + \alpha(\mathbf{Z}) \cdot \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{w} = \alpha(\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}) \cdot \mathbf{w}. \end{aligned} \tag{10.3}$$

Formula (10.3) is a key to further analysis of the structure of curvature tensor $\mathbf{R}$.

Theorem 10.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions for $n \geqslant 4$ there is a formula*

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \sigma(\mathbf{X}, \mathbf{Y}) \cdot \mathbf{Z} + \beta(\mathbf{Y}, \mathbf{Z}) \cdot \mathbf{X} - \beta(\mathbf{X}, \mathbf{Z}) \cdot \mathbf{Y} \tag{10.4}$$

expressing curvature tensor through two tensor fields $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$ of type $(0, 2)$*.*

For three arbitrary vectors $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ in (10.3) let's consider their linear span $U = \langle \mathbf{X}, \mathbf{Y}, \mathbf{Z} \rangle$. Due to $\dim V = n \geqslant 4$ subspace U do not coincide with V. Denote by $U^{\perp}$ the set of linear functionals α such that $\alpha(\mathbf{X}) = 0$, $\alpha(\mathbf{Y}) = 0$, and $\alpha(\mathbf{Z}) = 0$ simultaneously. Then $U^{\perp}$ is a subspace in dual space V^* such that $\dim U^{\perp} = n - \dim U$ (see for instance [18]) and

$$U = \{\mathbf{u} \in V : \quad \alpha(\mathbf{u}) = 0 \ \text{ for all } \ \alpha \in U^{\perp}\}. \tag{10.5}$$

Recall that α in (10.3) is an arbitrary linear functional. Substituting various linear functionals $\alpha \in U^{\perp}$ into (10.3) we find that $\alpha(\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}) = 0$ for them. Due to (10.5) this means that for any three vectors $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ vector $\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}$ is in their linear span. Let's express this circumstance as

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \beta \cdot \mathbf{X} + \gamma \cdot \mathbf{Y} + \sigma \cdot \mathbf{Z}. \tag{10.6}$$

Now denote by $U = \langle \mathbf{X}, \mathbf{Y} \rangle$ linear span of two vectors $\mathbf{X}$ and $\mathbf{Y}$. If $\mathbf{Z} \notin U$, then coefficient σ is uniquely defined by the expansion (10.6). Thus we have a function $\sigma = \sigma(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ defined for triples of vectors such that $\mathbf{Z} \notin \langle \mathbf{X}, \mathbf{Y} \rangle$. Further proof of theorem 10.1 breaks into series of lemmas.

Lemma 10.1. *For $\dim V \geqslant 4$ the function $\sigma = \sigma(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ doesn't depend on $\mathbf{Z}$.*

Proof. Let's retain the notation $U = \langle \mathbf{X}, \mathbf{Y} \rangle$ for linear span of the vectors $\mathbf{X}$ and $\mathbf{Y}$ and consider a factorspace V/U. Let $\mathbf{Z}_1$ and $\mathbf{Z}_2$ be two arbitrary vectors such that their cosets relative to subspace U are linearly independent. The existence of such vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ follows from the estimate

$$\dim(V/U) \geqslant 4 - 2 = 2.$$

For these two vectors we have $\mathbf{Z}_1 \notin U$ and $\mathbf{Z}_2 \notin U$ so that if $\mathbf{Z}_3 = \mathbf{Z}_1 + \mathbf{Z}_2$, then

$\mathbf{Z}_3 \notin U$. Let's write the equation (10.6) for each of three triples of vectors:

$$\begin{aligned} \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}_1 &= \beta_1 \cdot \mathbf{X} + \gamma_1 \cdot \mathbf{Y} + \sigma_1 \cdot \mathbf{Z}_1,\\ \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}_2 &= \beta_2 \cdot \mathbf{X} + \gamma_2 \cdot \mathbf{Y} + \sigma_2 \cdot \mathbf{Z}_2,\\ \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}_3 &= \beta_3 \cdot \mathbf{X} + \gamma_3 \cdot \mathbf{Y} + \sigma_3 \cdot \mathbf{Z}_3. \end{aligned} \tag{10.7}$$

Let's add first two equalities (10.7) and subtract the third one. Then factorize the obtained equality with respect to the subspace U. As a result we have

$$(\sigma_1 - \sigma_3) \cdot \operatorname{Cl}_U(\mathbf{Z}_1) + (\sigma_2 - \sigma_3) \cdot \operatorname{Cl}_U(\mathbf{Z}_2) = 0.$$

Since cosets $\operatorname{Cl}_U(\mathbf{Z}_1)$ and $\operatorname{Cl}_U(\mathbf{Z}_2)$ are linearly independent, from the above equality we obtain $\sigma_1 = \sigma_3$ and $\sigma_2 = \sigma_3$, where $\sigma_1 = \sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_1)$ and $\sigma_2 = \sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_2)$. Thus we have proved the required result $\sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_1) = \sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_2)$ for the vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ whose cosets are linearly independent.

Now suppose that cosets $\operatorname{Cl}_U(\mathbf{Z}_1)$ and $\operatorname{Cl}_U(\mathbf{Z}_1)$ are linearly dependent, but are nonzero. Then due to $\dim(V/U) \geqslant 2$ we can find a vector $\mathbf{Z}_4$ coset of which is not collinear to cosets $\operatorname{Cl}_U(\mathbf{Z}_1)$ and $\operatorname{Cl}_U(\mathbf{Z}_2)$, and we can apply previous result:

$$\sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_1) = \sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_4) = \sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_2).$$

Cases when $\operatorname{Cl}_U(\mathbf{Z}_1) = 0$ or $\operatorname{Cl}_U(\mathbf{Z}_2) = 0$ are not considered since in these cases the equality (10.6) do not define both quantities $\sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_1)$ and $\sigma(\mathbf{X},\mathbf{Y},\mathbf{Z}_2)$. $\square$

Lemma 10.1 shows that for $n \geqslant 4$ the equality (10.6) defines the function $\sigma = \sigma(\mathbf{X},\mathbf{Y})$ which do not depend on third vector $\mathbf{Z}$. This circumstance gives us the opportunity to expand (10.6) for the case when $\mathbf{Z}$ belongs to $U = \langle \mathbf{X},\mathbf{Y} \rangle$.

Lemma 10.2. *For* $\dim V \geqslant 4$ *the function* $\sigma(\mathbf{X},\mathbf{Y})$ *defined by the relationship* (10.6) *is linear in its first argument* $\mathbf{X}$.

Proof. Let $\mathbf{X}_1 + \mathbf{X}_2 = \mathbf{X}_3$. Denote by $\mathbf{U} = \langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y} \rangle$ linear span of four vectors $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, and $\mathbf{Y}$. Its obvious that $\dim U \leqslant 3$. Due to $\dim V \geqslant 4$ we can find a vector $\mathbf{Z} \notin U$. Hence $\operatorname{Cl}_U(\mathbf{Z}) \neq 0$, and we can write (10.6) three times:

$$\begin{aligned} \mathbf{R}(\mathbf{X}_1,\mathbf{Y})\mathbf{Z} &= \beta_1 \cdot \mathbf{X}_1 + \gamma_1 \cdot \mathbf{Y} + \sigma(\mathbf{X}_1,\mathbf{Y}) \cdot \mathbf{Z},\\ \mathbf{R}(\mathbf{X}_2,\mathbf{Y})\mathbf{Z} &= \beta_2 \cdot \mathbf{X}_2 + \gamma_2 \cdot \mathbf{Y} + \sigma(\mathbf{X}_2,\mathbf{Y}) \cdot \mathbf{Z},\\ \mathbf{R}(\mathbf{X}_3,\mathbf{Y})\mathbf{Z} &= \beta_3 \cdot \mathbf{X}_3 + \gamma_3 \cdot \mathbf{Y} + \sigma(\mathbf{X}_2,\mathbf{Y}) \cdot \mathbf{Z}. \end{aligned} \tag{10.8}$$

Let's add first two equalities (10.8) and subtract the third one. Then factorize with respect to the subspace U. As a result we have

$$\sigma(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{Y}) = \sigma(\mathbf{X}_1,\mathbf{Y}) + \sigma(\mathbf{X}_2,\mathbf{Y}). \tag{10.9}$$

Now suppose that we are given a real number $\alpha \in \mathbb{R}$ and two vectors $\mathbf{X}_1$ and $\mathbf{Y}$. Suppose $\mathbf{X}_2 = \alpha \cdot \mathbf{X}_1$ and consider linear span $U = \langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y} \rangle$ for the vectors $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{Y}$. Due to inequalities $\dim V \geqslant 4$ and $\dim U \leqslant 2$ we can find a vector $\mathbf{Z} \notin U$. Then write the equality (10.6) twice:

$$\begin{aligned} &\mathbf{R}(\mathbf{X}_1, \mathbf{Y})\mathbf{Z} = \beta_1 \cdot \mathbf{X}_1 + \gamma_2 \cdot \mathbf{Y} + \sigma(\mathbf{X}_1, \mathbf{Y}) \cdot \mathbf{Z}, \\ &\mathbf{R}(\mathbf{X}_2, \mathbf{Y})\mathbf{Z} = \beta_2 \cdot \mathbf{X}_2 + \gamma_2 \cdot \mathbf{Y} + \sigma(\mathbf{X}_2, \mathbf{Y}) \cdot \mathbf{Z}. \end{aligned} \tag{10.10}$$

Multiply first equality (10.10) by α and subtract second one from it. After factorizing with respect to the subspace U we finally get

$$\sigma(\alpha \cdot \mathbf{X}_1, \mathbf{Y}) = \alpha\, \sigma(\mathbf{X}_1, \mathbf{Y}).$$

This equality together with (10.9) proves linearity of $\sigma(\mathbf{X}, \mathbf{Y})$ in its first argument. Lemma is proved. $\square$

Lemma 10.3. *For* $\dim V \geqslant 4$ *the function* $\sigma(\mathbf{X}, \mathbf{Y})$ *defined by the relationship* (10.6) *is linear in its second argument* $\mathbf{Y}$*.*

Proof of the lemma 10.3 is quite similar to the proof of previous lemma 10.2 so that we can omit it. In whole, lemmas 10.1, 10.2, and 10.3 mean that the relationship (10.6) defines a bilinear form $\sigma(\mathbf{X}, \mathbf{Y})$ and corresponding tensor $\boldsymbol{\sigma}$ of type $(0, 2)$. Therefore we rewrite (10.6) as

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} - \sigma(\mathbf{X}, \mathbf{Y}) \cdot \mathbf{Z} = \beta \cdot \mathbf{X} + \gamma \cdot \mathbf{Y}. \tag{10.11}$$

Note that for to state and prove lemmas 10.1, 10.2, and 10.3 we used only the linearity of left hand side of (10.6) with respect to $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$. Left had side of (10.11) is also linear in $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$. Repeating previous arguments we obtain two bilinear forms $\beta(\mathbf{Y}, \mathbf{Z})$ and $\gamma(\mathbf{X}, \mathbf{Z})$ such that (10.11) can be written as

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \sigma(\mathbf{X}, \mathbf{Y}) \cdot \mathbf{Z} + \beta(\mathbf{Y}, \mathbf{Z}) \cdot \mathbf{X} + \gamma(\mathbf{X}, \mathbf{Z}) \cdot \mathbf{Y}. \tag{10.12}$$

Left hand side of (10.12) is skew-symmetric respective to $\mathbf{X}$ and $\mathbf{Y}$. This yields $\boldsymbol{\gamma} = -\boldsymbol{\beta}$. Therefore we can write (10.12) in form of (10.4), thus completing proof of the theorem 10.1. Furthermore skew symmetry of $\mathbf{R}(\mathbf{X}, \mathbf{Y})$ in $\mathbf{X}$ and $\mathbf{Y}$ implies skew symmetry of bilinear form $\sigma(\mathbf{X}, \mathbf{Y})$.

Formula (10.4) gives us the opportunity to express the components of curvature tensor through the components of two tensor fields $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$:

$$R^k_{sij} = \sigma_{ij}\, \delta^k_s + \beta_{js}\, \delta^k_i - \beta_{is}\, \delta^k_j. \tag{10.13}$$

Now if we make contractions in (10.13) according to the formulas (4.11), we obtain the following expressions for the components of tensor $\mathbf{S}$ and Ricci tensor $\mathbf{R}$:

$$S_{ij} = n\, \sigma_{ij} - (\beta_{ij} - \beta_{ji}), \qquad R_{ij} = \sigma_{ij} + (n - 1)\, \beta_{ji}.$$

Let $\hat{\boldsymbol\beta}$ and $\tilde{\boldsymbol\beta}$ be symmetric and skew-symmetric parts of tensor $\boldsymbol\beta$. Then

$$S_{ij} = n\,\sigma_{ij} - 2\,\tilde\beta_{ij}, \qquad \hat R_{ij} = (n-1)\,\hat\beta_{ij}, \qquad \tilde R_{ij} = \sigma_{ij} - (n-1)\,\tilde\beta_{ij}.$$

If we take into account the relationship $\mathbf{S} = 2\,\tilde{\mathbf{R}}$, we can express tensor fields $\boldsymbol\sigma$, $\hat{\boldsymbol\beta}$, and $\tilde{\boldsymbol\beta}$ through symmetric and skew-symmetric parts of Ricci tensor:

$$\hat{\boldsymbol\beta} = \frac{\hat{\mathbf{R}}}{n-1}, \qquad \tilde{\boldsymbol\beta} = -\frac{\tilde{\mathbf{R}}}{n+1}, \qquad \boldsymbol\sigma = \frac{2\,\tilde{\mathbf{R}}}{n+1}. \tag{10.14}$$

From (10.13) and (10.14) we derive formula for components of curvature tensor:

$$R^k_{sij} = \frac{2\,\tilde R_{ij}\,\delta^k_s - \tilde R_{js}\,\delta^k_i + \tilde R_{is}\,\delta^k_j}{n+1} + \frac{\hat R_{js}\,\delta^k_i - \hat R_{is}\,\delta^k_j}{n-1}. \tag{10.15}$$

Theorem 10.2. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions if $n \geqslant 4$, then appropriate curvature tensor is expressed through Ricci tensor according to the formula* (10.15)*.*

11. Structure of curvature tensor for $n = 3$.

Now we continue considering cases of intermediate degeneration when $1 \leqslant m \leqslant n-1$. Dimension $n = 3$ restricts our possibilities to maneuver in proving propositions like lemmas 10.1 and 10.2. In order to avoid these restrictions let's substitute $\mathbf{Z} = \mathbf{X}$ into the formula (10.3). This yields

$$\begin{aligned} &\alpha(\mathbf{X})\cdot\mathbf{R}(\mathbf{w},\mathbf{Y})\mathbf{X} + \alpha(\mathbf{Y})\cdot\mathbf{R}(\mathbf{X},\mathbf{w})\mathbf{X} + \\ &\quad + \alpha(\mathbf{X})\cdot\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{w} = \alpha(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X})\cdot\mathbf{w}. \end{aligned} \tag{11.1}$$

Theorem 11.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions if $n = 3$, then there is a relationship*

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X} = \theta(\mathbf{X},\mathbf{Y})\cdot\mathbf{X} - \beta(\mathbf{X},\mathbf{X})\cdot\mathbf{Y} \tag{11.2}$$

binding curvature tensor with two tensor fields $\boldsymbol\beta$ and $\boldsymbol\theta$ of type $(0,2)$.

For two arbitrary vectors $\mathbf{X}$ and $\mathbf{Y}$ we consider their linear span $U = \langle\mathbf{X},\mathbf{Y}\rangle$. Due to $\dim V = n = 3$ subspace U do not coincide with V. Denote by $U^\perp$ the set of linear functionals α such that $\alpha(\mathbf{X}) = 0$ and $\alpha(\mathbf{Y}) = 0$ simultaneously. Then $U^\perp$ is a subspace in dual space V^* such that

$$U = \{\mathbf{u} \in V : \quad \alpha(\mathbf{u}) = 0 \ \text{ for all } \ \alpha \in U^\perp\}. \tag{11.3}$$

Recall that α in (11.1) is an arbitrary linear functional. Substituting various linear functionals $\alpha \in U^{\perp}$ into (11.1) we find that $\alpha(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X}) = 0$ for them. Due to (11.3) this means that for any two vectors $\mathbf{X}$ and $\mathbf{Y}$ vector $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X}$ is in their linear span. Let's express this circumstance as

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X} = \theta \cdot \mathbf{X} - \beta \cdot \mathbf{Y}. \tag{11.4}$$

Now denote by $U = \langle \mathbf{X} \rangle$ linear span of the vector $\mathbf{X}$. If $\mathbf{Y} \notin U$, then coefficient β is uniquely defined by the expansion (11.4). Thus we have a function $\beta = \beta(\mathbf{X},\mathbf{Y})$ defined for pairs of vectors such that $\mathbf{Y} \notin \langle \mathbf{X} \rangle$. Further proof of theorem 11.1 breaks into series of lemmas.

Lemma 11.1. *For $n = 3$ the function $\beta = \beta(\mathbf{X},\mathbf{Y})$ doesn't depend on $\mathbf{Y}$.*

Proof. Let's retain the notation $U = \langle \mathbf{X},\mathbf{Y} \rangle$ for linear span of the vector $\mathbf{X}$ and consider a factorspace V/U. Let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be two arbitrary vectors such that their cosets relative to subspace U are linearly independent. The existence of such vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$ follows from the estimate

$$\dim(V/U) \geqslant 3 - 1 = 2.$$

For these two vectors we have $\mathbf{Y}_1 \notin U$ and $\mathbf{Y}_2 \notin U$ so that if $\mathbf{Y}_3 = \mathbf{Y}_1 + \mathbf{Y}_2$, then $\mathbf{Y}_3 \notin U$. Let's write the equation (11.4) for each of three pairs of vectors:

$$\begin{aligned}
&\mathbf{R}(\mathbf{X},\mathbf{Y}_1)\mathbf{X} = \theta_1 \cdot \mathbf{X} - \beta_1 \cdot \mathbf{Y}_1,\\
&\mathbf{R}(\mathbf{X},\mathbf{Y}_2)\mathbf{X} = \theta_2 \cdot \mathbf{X} - \beta_2 \cdot \mathbf{Y}_2,\\
&\mathbf{R}(\mathbf{X},\mathbf{Y}_3)\mathbf{X} = \theta_3 \cdot \mathbf{X} - \beta_3 \cdot \mathbf{Y}_3.
\end{aligned} \tag{11.5}$$

Let's add first two equalities (11.5) and subtract the third one. Then factorize the obtained equality with respect to the subspace U. As a result we have

$$(\beta_1 - \beta_3) \cdot \mathrm{Cl}_U(\mathbf{Y}_1) + (\beta_2 - \beta_3) \cdot \mathrm{Cl}_U(\mathbf{Y}_2) = 0.$$

Since cosets $\mathrm{Cl}_U(\mathbf{Y}_1)$ and $\mathrm{Cl}_U(\mathbf{Y}_2)$ are linearly independent, from the above equality we obtain $\beta_1 = \beta_3$ and $\beta_2 = \beta_3$, where $\beta_1 = \beta(\mathbf{X},\mathbf{Y}_1)$ and $\beta_2 = \beta(\mathbf{X},\mathbf{Y}_2)$. Thus we have proved the required result $\beta(\mathbf{X},\mathbf{Y}_1) = \beta(\mathbf{X},\mathbf{Y}_2)$ for the vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2$, whose cosets are linearly independent.

Now suppose that cosets $\mathrm{Cl}_U(\mathbf{Y}_1)$ and $\mathrm{Cl}_U(\mathbf{Y}_1)$ are linearly dependent, but are nonzero. Then due to $\dim(V/U) \geqslant 2$ we can find a vector $\mathbf{Y}_4$ coset of which is not collinear to cosets $\mathrm{Cl}_U(\mathbf{Y}_1)$ and $\mathrm{Cl}_U(\mathbf{Y}_2)$, and we can apply previous result:

$$\beta(\mathbf{X},\mathbf{Y}_1) = \beta(\mathbf{X},\mathbf{Y}_4) = \beta(\mathbf{X},\mathbf{Y}_2).$$

Cases when $\mathrm{Cl}_U(\mathbf{Y}_1) = 0$ or $\mathrm{Cl}_U(\mathbf{Y}_2) = 0$ are not considered since in these cases the equality (11.4) do not define both quantities $\beta(\mathbf{X},\mathbf{Y}_1)$ and $\beta(\mathbf{X},\mathbf{Y}_2)$. $\square$

Lemma 11.1 shows that for the dimension $n = 3$ the equality (11.4) defines the function $\beta = \beta(\mathbf{X},\mathbf{Y})$, which do not depend on second vector $\mathbf{Y}$. This circumstance gives us the opportunity to expand (11.4) for the case when $\mathbf{Y}$ belongs to linear span of vector $\mathbf{X}$.

Remember that numerical function $\beta(\mathbf{X})$ of one vectorial argument is called quadratic form if it is obtained from some bilinear form $\gamma(\mathbf{X},\mathbf{Z})$ by substitution $\mathbf{Z}=\mathbf{X}$, i. e. $\beta(\mathbf{X})=\gamma(\mathbf{X},\mathbf{X})$. Without loss of generality bilinear form γ can be assumed to be symmetric (see more details in [18]).

Lemma 11.2. *Numerical function of one vectorial argument $\beta(\mathbf{X})$ is a quadratic form if and only if it satisfies the relationship*

$$\beta(\mathbf{X}+\alpha\cdot\mathbf{Z})+\alpha\,\beta(\mathbf{X}-\mathbf{Z})=(1+\alpha)\,\beta(\mathbf{X})+(\alpha+\alpha^2)\,\beta(\mathbf{Z}),\tag{11.6}$$

where $\mathbf{X}$ and $\mathbf{Z}$ are two arbitrary vectors and α is an arbitrary number.

Proof. Direct statement of lemma is obvious. Indeed, if for some bilinear form γ we substitute $\beta(\mathbf{X})=\gamma(\mathbf{X},\mathbf{X})$ into the relationship (11.6), we find that it is fulfilled identically.

Now let's prove converse assertion. Suppose (11.6) to be fulfilled for the function $\beta(\mathbf{X})$. Substituting $\mathbf{X}=\mathbf{Z}=0$ and $\alpha=1$ into (11.6) we get

$$\beta(0)=0.\tag{11.7}$$

As a second step we substitute $\mathbf{Z}=\mathbf{X}$ into (11.6). If we take into account (11.7), we obtain $\beta((1+\alpha)\cdot\mathbf{X})=(1+\alpha)^2\,\beta(\mathbf{X})$. This equality can be simplified by substituting $\alpha-1$ for α. It takes the following form:

$$\beta(\alpha\cdot\mathbf{X})=\alpha^2\,\beta(\mathbf{X}).\tag{11.8}$$

So β is a homogeneous function of degree 2. Now we use $\beta(\mathbf{X})$ in order to define the function $\gamma(\mathbf{X},\mathbf{Z})$ of two vectorial arguments:

$$\gamma(\mathbf{X},\mathbf{Z})=\frac{\beta(\mathbf{X}+\mathbf{Z})-\beta(\mathbf{X}-\mathbf{Z})}{4}.\tag{11.9}$$

Due to (11.8) the function $\gamma(\mathbf{X},\mathbf{Z})$ in (11.9) is symmetric: $\gamma(\mathbf{X},\mathbf{Z})=\gamma(\mathbf{Z},\mathbf{X})$. Let's prove that it is linear respective to its second argument. We write the relationship (11.6) substituting α for $-\alpha$ in it:

$$\beta(\mathbf{X}-\alpha\cdot\mathbf{Z})-\alpha\,\beta(\mathbf{X}-\mathbf{Z})=(1-\alpha)\,\beta(\mathbf{X})+(\alpha^2-\alpha)\,\beta(\mathbf{Z}).\tag{11.10}$$

Let's subtract (11.10) from the initial relationship (11.6). This yields

$$4\,\gamma(\mathbf{X},\alpha\cdot\mathbf{Z})=\beta(\mathbf{X}+\alpha\cdot\mathbf{Z})-\beta(\mathbf{X}-\alpha\cdot\mathbf{Z})=2\,\alpha\,(\beta(\mathbf{X})+\beta(\mathbf{Z})-\beta(\mathbf{X}-\mathbf{Z})).$$

For $\alpha=1$ this relationship takes the form

$$4\,\gamma(\mathbf{X},\mathbf{Z})=\beta(\mathbf{X}+\mathbf{Z})-\beta(\mathbf{X}-\mathbf{Z})=2\,(\beta(\mathbf{X})+\beta(\mathbf{Z})-\beta(\mathbf{X}-\mathbf{Z})).$$

Comparing two above relationships we find that

$$\gamma(\mathbf{X}, \alpha \cdot \mathbf{Z}) = \alpha\, \gamma(\mathbf{X}, \mathbf{Z}). \tag{11.11}$$

For $\alpha = 1$ the relationship (11.6) can be written as follows:

$$2\,\beta(\mathbf{X}) + 2\,\beta(\mathbf{Z}) = \beta(\mathbf{X} + \mathbf{Z}) + \beta(\mathbf{X} - \mathbf{Z}). \tag{11.12}$$

Let's substitute $\mathbf{X}$ by $\mathbf{X} + \mathbf{Z}_1$ and $\mathbf{Z}$ by $\mathbf{X} + \mathbf{Z}_2$ in (11.12). As a result we obtain

$$2\,\beta(\mathbf{X} + \mathbf{Z}_1) + 2\,\beta(\mathbf{X} + \mathbf{Z}_2) = \beta(2 \cdot \mathbf{X} + \mathbf{Z}_1 + \mathbf{Z}_2) + \beta(\mathbf{Z}_1 - \mathbf{Z}_2). \tag{11.13}$$

Further in formula (11.12) we substitute $\mathbf{X}$ by $\mathbf{X} - \mathbf{Z}_1$ and $\mathbf{Z}$ by $\mathbf{X} - \mathbf{Z}_2$:

$$2\,\beta(\mathbf{X} - \mathbf{Z}_1) + 2\,\beta(\mathbf{X} - \mathbf{Z}_2) = \beta(2 \cdot \mathbf{X} - \mathbf{Z}_1 - \mathbf{Z}_2) + \beta(\mathbf{Z}_1 - \mathbf{Z}_2). \tag{11.14}$$

Then we subtract (11.14) from (11.13) and compare the expressions in both sides of the obtained formula with (11.9). As a result of this comparison we find

$$8\,\gamma(\mathbf{X}, \mathbf{Z}_1) + 8\,\gamma(\mathbf{X}, \mathbf{Z}_2) = 4\,\gamma(2 \cdot \mathbf{X}, \mathbf{Z}_1 + \mathbf{Z}_2).$$

If we take into account (11.11) and the symmetry of the function $\gamma(\mathbf{X}, \mathbf{Z})$, we can bring this relationship to the following form:

$$\gamma(\mathbf{X}, \mathbf{Z}_1) + \gamma(\mathbf{X}, \mathbf{Z}_2) = \gamma(\mathbf{X}, \mathbf{Z}_1 + \mathbf{Z}_2). \tag{11.15}$$

The relationships (11.11) and (11.15) in the aggregate mean the linearity of $\gamma(\mathbf{X}, \mathbf{Z})$ respective to $\mathbf{Z}$. Linearity of $\gamma(\mathbf{X}, \mathbf{Z})$ respective to $\mathbf{X}$ then follows by the symmetry $\gamma(\mathbf{X}, \mathbf{Z}) = \gamma(\mathbf{Z}, \mathbf{X})$. Thus we constructed symmetric bilinear form $\gamma(\mathbf{X}, \mathbf{Z})$. Due to (11.7) and (11.8) if we substitute $\mathbf{Z} = \mathbf{X}$ in $\gamma(\mathbf{X}, \mathbf{Z})$, we get $\beta(\mathbf{X}) = \gamma(\mathbf{X}, \mathbf{X})$. Proof is complete. $\square$

Now let's return to the formula (11.4). Due to the lemma 11.1 proved above this formula can be written as:

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{X} = \theta \cdot \mathbf{X} - \beta(\mathbf{X}) \cdot \mathbf{Y}. \tag{11.16}$$

Left hand side of (11.16) satisfies the relationship

$$\begin{gathered} \mathbf{R}(\mathbf{X} + \alpha \cdot \mathbf{Z}, \mathbf{Y})(\mathbf{X} + \alpha \cdot \mathbf{Z}) + \alpha\, \mathbf{R}(\mathbf{X} + \mathbf{Z}, \mathbf{Y})(\mathbf{X} + \mathbf{Z}) = \\ = (1 + \alpha)\, \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{X} + (\alpha + \alpha^2)\, \mathbf{R}(\mathbf{Z}, \mathbf{Y})\mathbf{Z}, \end{gathered} \tag{11.17}$$

which can be checked by direct calculations. For the case $\dim V = 3$ we can choose vector $\mathbf{Y}$ such that it doesn't belong to the linear span of $\mathbf{X}$ and $\mathbf{Z}$. Therefore, if we transform each summand in (11.17) by means of (11.16), we obtain vectorial equality which leads to the relationship (11.6) for the function $\beta(\mathbf{X})$.

Lemma 11.3. *For the dimension $n = 3$ function $\beta(\mathbf{X})$ in* (11.16) *is a quadratic form, which is defined by some bilinear form.*

Lemma 11.3 is an immediate consequence of the lemma 11.2 due to above arguments. We express the result of this lemma as follows:

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X} + \beta(\mathbf{X},\mathbf{X}) \cdot \mathbf{Y} = \theta \cdot \mathbf{X}. \tag{11.18}$$

Lemma 11.4. *For the dimension $n = 3$ the coefficient θ in* (11.18) *is a bilinear function of two vectorial arguments $\mathbf{X}$ and $\mathbf{Y}$.*

Proof. In order to prove the linearity o $\theta(\mathbf{X},\mathbf{Y})$ with respect to $\mathbf{Y}$ we should note that for any fixed vector $\mathbf{X} \neq 0$ in the space V of dimension $n = 3$ one can find linear functional γ such that $\gamma(\mathbf{X}) = 1$. Applying this functional to both sides of the equality (11.18) we obtain

$$\theta(\mathbf{X},\mathbf{Y}) = \gamma(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X}) + \beta(\mathbf{X},\mathbf{X})\,\gamma(\mathbf{Y}).$$

Linearity of $\theta(\mathbf{X},\mathbf{Y})$ respective to $\mathbf{Y}$ now is a consequence of linearity of right hand side of the above equality respective to $\mathbf{Y}$ for fixed γ and $\mathbf{X}$.

Let's prove linearity of $\theta(\mathbf{X},\mathbf{Y})$ respective to $\mathbf{X}$. Suppose $\mathbf{X} \neq 0$. If we substitute $\alpha \cdot \mathbf{X}$ for $\mathbf{X}$ into the formula (11.18), we get

$$\alpha^2 \cdot \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X} + \alpha^2\,\beta(\mathbf{X},\mathbf{X}) \cdot \mathbf{Y} = \alpha\,\theta(\alpha \cdot \mathbf{X},\mathbf{Y}) \cdot \mathbf{X}.$$

Comparing this with the equality $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X} + \beta(\mathbf{X},\mathbf{X}) \cdot \mathbf{Y} = \theta(\mathbf{X},\mathbf{Y}) \cdot \mathbf{X}$ we find

$$\theta(\alpha \cdot \mathbf{X},\mathbf{Y}) = \alpha\,\theta(\mathbf{X},\mathbf{Y}). \tag{11.19}$$

For $\mathbf{X} = 0$ the quantity θ is not correctly defined from (11.18). The relationship (11.19) makes possible to extend the function $\theta(\mathbf{X},\mathbf{Y})$ for $\mathbf{X} = 0$ by the value $\theta(0,\mathbf{Y}) = 0$.

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two arbitrary vectors. Denote $\mathbf{X}_3 = \mathbf{X}_1 + \mathbf{X}_2$. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are collinear, then there is a vector $\mathbf{e} \neq 0$ such that

$$\mathbf{X}_1 = \alpha_1 \cdot \mathbf{e}, \qquad \mathbf{X}_2 = \alpha_2 \cdot \mathbf{e}, \qquad \mathbf{X}_3 = \alpha_3 \cdot \mathbf{e},$$

where $\alpha_3 = \alpha_1 + \alpha_2$. By means of (11.19) we obtain $\theta(\mathbf{X}_1,\mathbf{Y}) + \theta(\mathbf{X}_2,\mathbf{Y}) = \alpha_1\,\theta(\mathbf{e},\mathbf{Y}) + \alpha_2\,\theta(\mathbf{e},\mathbf{Y}) = \alpha_3\,\theta(\mathbf{e},\mathbf{Y}) = \theta(\mathbf{X}_3,\mathbf{Y})$. This means

$$\theta(\mathbf{X}_1,\mathbf{Y}) + \theta(\mathbf{X}_2,\mathbf{Y}) = \theta(\mathbf{X}_1 + \mathbf{X}_2,\mathbf{Y}). \tag{11.20}$$

Now suppose that vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are not collinear. Let γ be an arbitrary linear functional in V. Let's apply it to both sides of (11.18). As a result we get

$$\gamma(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X}) + \beta(\mathbf{X},\mathbf{X})\,\gamma(\mathbf{Y}) = \theta(\mathbf{X},\mathbf{Y})\,\gamma(\mathbf{X}). \tag{11.21}$$

For the fixed vector $\mathbf{Y}$ left hand side of (11.21) is a quadratic form respective to $\mathbf{X}$ (it is obtained from bilinear function $\gamma(\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z})+\beta(\mathbf{X},\mathbf{Z})\,\gamma(\mathbf{Y})$ by substitution $\mathbf{Z}=\mathbf{X}$). Therefore right hand side of (11.21) is also a quadratic form respective to $\mathbf{X}$. According to the lemma 11.2 it satisfies the relationship (11.6) where we can take $\alpha=1$, $\mathbf{X}=\mathbf{X}_1$, and $\mathbf{Z}=\mathbf{X}_2$. This yields

$$\begin{gathered}\theta(\mathbf{X}_1+\mathbf{X}_2,\mathbf{Y})\,\gamma(\mathbf{X}_1+\mathbf{X}_2)+\theta(\mathbf{X}_1-\mathbf{X}_2,\mathbf{Y})\,\gamma(\mathbf{X}_1-\mathbf{X}_2)=\\ 2\,\theta(\mathbf{X}_1,\mathbf{Y})\,\gamma(\mathbf{X}_1)+2\,\theta(\mathbf{X}_2,\mathbf{Y})\,\gamma(\mathbf{X}_2).\end{gathered}\tag{11.22}$$

Relying on the linearity of functional γ we can bring (11.22) to the form

$$\begin{gathered}(\theta(\mathbf{X}_1+\mathbf{X}_2,\mathbf{Y})+\theta(\mathbf{X}_1-\mathbf{X}_2,\mathbf{Y})-2\,\theta(\mathbf{X}_1,\mathbf{Y}))\,\gamma(\mathbf{X}_1)+\\ +(\theta(\mathbf{X}_1+\mathbf{X}_2,\mathbf{Y})-\theta(\mathbf{X}_1-\mathbf{X}_2,\mathbf{Y})-2\,\theta(\mathbf{X}_2,\mathbf{Y}))\,\gamma(\mathbf{X}_2)=0.\end{gathered}\tag{11.23}$$

Functional γ in (11.23) is an arbitrary linear functional. For the pair of noncollinear vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ we can find pair of functionals γ_2 and γ_2 such that

$$\gamma_1(\mathbf{X}_1)=1,\qquad\gamma_1(\mathbf{X}_2)=0,\qquad\gamma_2(\mathbf{X}_1)=0,\qquad\gamma_2(\mathbf{X}_2)=1.$$

Therefore the relationship (11.23) can be broken into two separate parts:

$$\begin{aligned}&\theta(\mathbf{X}_1+\mathbf{X}_2,\mathbf{Y})+\theta(\mathbf{X}_1-\mathbf{X}_2,\mathbf{Y})=2\,\theta(\mathbf{X}_1,\mathbf{Y}),\\ &\theta(\mathbf{X}_1+\mathbf{X}_2,\mathbf{Y})-\theta(\mathbf{X}_1-\mathbf{X}_2,\mathbf{Y})=2\,\theta(\mathbf{X}_2,\mathbf{Y}).\end{aligned}$$

If we add these two equalities and divide the result by 2, we come to the relationship (11.20), which appears to be fulfilled for noncollinear vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ as well. In the aggregate with (11.19) it means that $\theta(\mathbf{X},\mathbf{Y})$ is linear respective to its first argument $\mathbf{X}$. Lemma is proved. $\square$

Lemma 11.4 completes the proof of the theorem 11.1. We shall use the equality (11.2) from this theorem for the further calculations. Let's transform quadratic in $\mathbf{X}$ expression $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{X}$ into bilinear one respective to $\mathbf{X}$ and $\mathbf{Z}$:

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}+\mathbf{R}(\mathbf{Z},\mathbf{Y})\mathbf{X}=\frac{\mathbf{R}(\mathbf{X}+\mathbf{Z},\mathbf{Y})(\mathbf{X}+\mathbf{Z})-\mathbf{R}(\mathbf{X}-\mathbf{Z},\mathbf{Y})(\mathbf{X}-\mathbf{Z})}{2}.$$

Applying formula (11.2) we bring this relationship to the form

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}+\mathbf{R}(\mathbf{Z},\mathbf{Y})\mathbf{X}=\theta(\mathbf{X},\mathbf{Y})\cdot\mathbf{Z}+\theta(\mathbf{Z},\mathbf{Y})\cdot\mathbf{X}-2\,\beta(\mathbf{X},\mathbf{Z})\cdot\mathbf{Y}.$$

Here $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are arbitrary vectors. Therefore we can get the following relationship for the components of the curvature tensor:

$$R^k_{sij}+R^k_{isj}=\theta_{ij}\,\delta^k_s+\theta_{sj}\,\delta^k_i-2\,\beta_{is}\,\delta^k_j.\tag{11.24}$$

Let's alternate (11.24) with respect to the pair of indices i and j:

$$R^k_{sij}+\frac{R^k_{isj}-R^k_{jsi}}{2}=\tilde\theta_{ij}\,\delta^k_s+\frac{\theta_{sj}-2\,\beta_{sj}}{2}\,\delta^k_i-\frac{\theta_{si}-2\,\beta_{si}}{2}\,\delta^k_j. \tag{11.25}$$

Here $\tilde{\boldsymbol\theta}$ is skew-symmetric part of tensor $\boldsymbol\theta$. For to transform the left hand side of (11.25) we use skew symmetry $R^k_{jsi}=-R^k_{jis}$, $R^k_{sji}=-R^k_{sij}$, and the identity $R^k_{sji}+R^k_{isj}+R^k_{jis}=0$, which follows from $\Gamma^k_{rs}=\Gamma^k_{sr}$:

$$\frac{3\,R^k_{sij}}{2}=\tilde\theta_{ij}\,\delta^k_s+\frac{\theta_{sj}-2\,\beta_{sj}}{2}\,\delta^k_i-\frac{\theta_{si}-2\,\beta_{si}}{2}\,\delta^k_j. \tag{11.26}$$

Now let's compare the formula (10.13) for the case $\dim V\geqslant 4$ with the formula (11.26) just derived for the case $\dim V=3$. Comparing we see that the structure of components of curvature tensor for three-dimensional case is the same as for the cases of higher dimension. Therefore formula (10.15) remains true for $n=3$ and we can state the following theorem similar to theorem 10.2.

Theorem 11.2. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions if $n=3$, then appropriate curvature tensor is expressed through Ricci tensor according to the formula* (10.15)*.*

12. Structure of curvature tensor for $n=2$.

For $n=2$ due to the inequalities $1\leqslant m\leqslant n-1$ we have only one case of intermediate degeneration with $m=1$. According to lemma 9.2 tensor $\mathbf S$ defined in (4.11) and skew-symmetric part of Ricci tensor $\tilde{\mathbf R}$ in this case are zero: $\mathbf S=2\,\tilde{\mathbf R}=0$. Curvature tensor is completely determined by symmetric part of Ricci tensor $\hat{\mathbf R}$:

$$R^k_{sij}=\hat R_{sj}\,\delta^k_i-\hat R_{si}\,\delta^k_j. \tag{12.1}$$

In order to prove formula (12.1) note that for $n=2$ each skew-symmetric in pair of indices numeric array is proportional to skew-symmetric unit matrix

$$d^{ij}=d_{ij}=\begin{Vmatrix}0&1\\-1&0\end{Vmatrix}. \tag{12.2}$$

When applied to curvature tensor, this yields $R^k_{sij}=\rho^k_s\,d_{ij}$. Hence we can calculate Ricci tensor, which is known to by symmetric by lemma 9.2:

$$\hat R_{sj}=R_{sj}=\sum^2_{k=1}\rho^k_s\,d_{kj}.$$

Matrix (12.2) is invertible. Therefore we can express ρ^k_s through Ricci tensor:

$$\rho^k_s=-\sum^2_{r=1}\hat R_{sr}\,d^{rk}. \tag{12.3}$$

Now we are to substitute (12.3) into the formula $R^k_{sij} = \rho^k_s\, d_{ij}$ and use well-known identity $d^{rk}\, d_{ij} = \delta^r_i\, \delta^k_j - \delta^k_i\, \delta^r_j$. As a result we will get (12.1).

13. Structure of tensor field A.

Let (1.1) be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension (9.1). Choose some fixed point p_0 on M and consider a field of point symmetry $\boldsymbol{\eta}$ for (1.1) vanishing at the point p_0. The condition $L_{\boldsymbol{\eta}}(\mathbf{A}) = 0$ from (2.5) written at the point p_0 for such symmetry field $\boldsymbol{\eta}$ has the following form:

$$\mathbf{A}\,\mathbf{F} = \mathbf{F}\,\mathbf{A}. \tag{13.1}$$

Here $\mathbf{F} = L_{\boldsymbol{\eta}}$ is linear operator from (5.7). Due to (9.1) any operator $\mathbf{F}$ satisfying (8.3) corresponds to some symmetry field $\boldsymbol{\eta} \in \mathcal{G}(p_0)$. In particular, we can choose $\mathbf{F} = \mathbf{w} \otimes \alpha$, where $\mathbf{w}$ is some arbitrary nonzero vector from the kernel of bilinear form $\hat{R}$ and α is arbitrary linear functional in $V = T_{p_0}(M)$. Substituting this operator into (13.1) we obtain the relationship

$$\alpha(\mathbf{X}) \cdot \mathbf{A}\mathbf{w} = \alpha(\mathbf{A}\mathbf{X}) \cdot \mathbf{w}, \tag{13.2}$$

where $\mathbf{X}$ is an arbitrary vector of V. For α in (13.2) we can choose linear functional such that $\alpha(\mathbf{X}) \neq 0$. Then the equality (13.2) takes the form

$$\mathbf{A}\mathbf{w} = a \cdot \mathbf{w},$$

where numeric parameter a is defined by formula $a = \alpha(\mathbf{A}\mathbf{X})/\alpha(\mathbf{X})$. Therefore a doesn't depend on the choice of $\mathbf{w} \in \operatorname{Ker} \hat{R}$. Vector $\mathbf{w} \neq 0$ is an eigenvector for the operator $\mathbf{A}$ corresponding to the eigenvalue a. Hence a do not depend on the choice of $\mathbf{X}$ and α in (13.2). From $\mathbf{A}\mathbf{w} = a \cdot \mathbf{w}$ and (13.2) we get $\alpha(\mathbf{A}\mathbf{X}) = \alpha(a \cdot \mathbf{X})$. Since α is arbitrary linear functional, we conclude

$$\mathbf{A}\mathbf{X} = a \cdot \mathbf{X}. \tag{13.3}$$

The equality (13.3) fulfilled for arbitrary vector $\mathbf{X} \in V$ means that $\mathbf{A}$ is a scalar operator. It differs from identical operator $\mathbf{id}_V$ by a scalar factor: $\mathbf{A} = a \cdot \mathbf{id}_V$.

Let's substitute $\mathbf{A} = a \cdot \mathbf{id}_V$ into the equation (2.5), where point symmetry field $\boldsymbol{\eta}$ is no longer vanishing at the point p_0:

$$L_{\boldsymbol{\eta}}(\mathbf{A}) = L_{\boldsymbol{\eta}}(a) \cdot \mathbf{id}_V = 0. \tag{13.4}$$

Therefore $L_{\boldsymbol{\eta}}(a) = 0$. From (9.1) and the estimates (8.1) and (8.9) we get

$$\dim(\mathcal{G}/\mathcal{G}(p_0)) = \dim M = n.$$

Hence for each point $p_0 \in M$ we can find n vector fields from $\mathcal{G}$ whose values at this point are linear independent. Therefore $L_{\boldsymbol{\eta}}(a) = 0$ implies $a = \text{const}$.

Theorem 13.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Then A is a constant scalar matrix, i. e. $A^i_j = a\,\delta^i_j$ and $a = \text{const}$.*

14. Projectively-euclidean spaces.

For the systems of equations (1.1) possessing symmetry algebras of maximal order (9.1) matrices A in (1.1) are the same in all cases of intermediate degeneration: $A^i_j = a\,\delta^i_j$. Therefore structure of such systems of equations are completely defined by components of affine connection Γ^j_{rs} in (1.1).

Let $\mathbf{R}$ be curvature tensor corresponding to the connection Γ in (1.1). According to theorems 10.2 and 11.2 for $n \geqslant 3$ its components are given by formula (10.15). In order to simplify notations let's use formula (10.13) instead of (10.15). Tensor fields $\boldsymbol\beta$ and $\boldsymbol\sigma$ in (10.13) are expressed through Ricci tensor according to (10.14). Taking into account the relationship $\boldsymbol\sigma = -2\,\tilde{\boldsymbol\beta}$, which follows from (10.14), we can bring (10.13) to the following rather simple form:

$$R^k_{sij} = (\beta_{ji} - \beta_{ij})\,\delta^k_s + \beta_{js}\,\delta^k_i - \beta_{is}\,\delta^k_j. \tag{14.1}$$

Let's substitute (14.1) into the well-known Bianchi-Padov identity (see [17], [22]):

$$\nabla_i R^k_{sjq} + \nabla_j R^k_{sqi} + \nabla_q R^k_{sij} = 0.$$

If $n \geqslant 3$, this substitution results in the equation for tensor field $\boldsymbol\beta$:

$$\nabla_i \beta_{js} = \nabla_j \beta_{is}. \tag{14.2}$$

According to [22] (see chapter 4, § 47) we write the following differential equation for some covector field $\mathbf{u}$:

$$\nabla_i u_j = \beta_{ij} + u_i\,u_j, \quad i, j = 1, \ldots, n. \tag{14.3}$$

If we express covariant derivatives through partial derivatives, we see that (14.3) is a complete system of Pfaff equations:

$$\frac{\partial u_j}{\partial y^i} = \beta_{ij} + u_i\,u_j + \sum_{k=1}^{n} \Gamma^k_{ij}\,u_k. \tag{14.4}$$

By direct calculations one can find that the relationships (14.1) and (14.2) provide complete compatibility for Pfaff equations (14.4). This in turn means that for differential equations (14.3) the Cauchy problem

$$u_j\,\Big|_{p=p_0} = u_j(0) \tag{14.5}$$

is solvable in some neighborhood of the point $p = p_0$ in M for arbitrary initial values $u_j(0)$ in (14.5).

Let $\mathbf{u}$ be a covector field obtained by solving Cauchy problem (14.5) for the equations (14.3). We shall use this field to construct another connection $\bar{\Gamma}$:

$$\bar{\Gamma}^k_{rs} = \Gamma^k_{rs} + u_r\,\delta^k_s + u_s\,\delta^k_r. \tag{14.6}$$

Let's calculate curvature tensor of new connection $\bar{\Gamma}$ on the base of formula (2.8). Taking in to account (14.1) and (14.3) we get $\bar{R}^k_{sij} = 0$. This means that (14.1) is flat (euclidean) affine connection.

Transformation of connection components $\bar{\Gamma}^k_{rs} \to \Gamma^k_{rs} = \bar{\Gamma}^k_{rs} - u_r\,\delta^k_s - u_s\,\delta^k_r$ is called *projective transformation* (see [22]). Affine connection Γ obtained from euclidean connection $\bar{\Gamma}$ by this transformation is called *projectively-euclidean connection.* Manifolds equipped with projectively-euclidean connection are called *projectively-euclidean spaces.* For euclidean connection (14.6) one can find local coordinates $y^1, \ldots, y^n$ such that $\bar{\Gamma}^k_{rs} = 0$. Relative to projectively-euclidean connection Γ these local coordinates are called *projectively-euclidean coordinates.* In projectively-euclidean coordinates for Γ^k_{rs} we have

$$\Gamma^k_{rs} = -(u_r\,\delta^k_s + u_s\,\delta^k_r). \tag{14.7}$$

Theorem 14.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions if $n \geqslant 3$, then there is a point transformation* (1.4) *bringing these equations to the form*

$$\frac{\partial y^i}{\partial t} = a\,\frac{\partial^2 y^i}{\partial x^2} - 2\,a\left(\sum_{r=1}^{n} u_r\,\frac{\partial y^r}{\partial x}\right)\frac{\partial y^i}{\partial x}, \quad i = 1, \ldots, n,$$

where $a = \mathrm{const}$ and $u_r = u_r(y^1, \ldots, y^n)$ are components of some covector field $\mathbf{u}$.

15. Structure of tensor $\mathbf{Q} = \nabla\mathbf{R}$.

Denote by $\mathbf{Q}$ covariant differential of Ricci tensor: $\mathbf{Q} = \nabla\mathbf{R}$ (in local coordinates $Q_{ijk} = \nabla_i R_{jk}$). $\mathbf{Q}$ is a tensor field of type $(0,3)$ belonging to the algebra $\mathcal{R}$ from (4.10). According to the theorem 4.3 we have

$$L_{\boldsymbol{\eta}}(\mathbf{Q}) = 0 \tag{15.1}$$

for any point symmetry field $\boldsymbol{\eta}$ of the system of equations (1.1). Let's take and fix some point $p = p_0$ on M. Tensor $\mathbf{Q}$ defines trilinear form on $V = T_{p_0}(M)$:

$$Q(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \nabla_i R_{jk}\,Z^i\,X^j\,Y^k. \tag{15.2}$$

From (15.1) we get the following equality for trilinear form (15.2):

$$Q(\mathbf{F}\mathbf{X},\mathbf{Y},\mathbf{Z}) + Q(\mathbf{X},\mathbf{F}\mathbf{Y},\mathbf{Z}) + Q(\mathbf{X},\mathbf{Y},\mathbf{F}\mathbf{Z}) = 0. \tag{15.3}$$

Here $\mathbf{F}$ is linear operator (5.7) defined by point symmetry field $\boldsymbol{\eta}$ vanishing at p_0.

Lemma 15.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions if $n \geqslant 3$, then*

$$Q(\mathbf{w}, \mathbf{Y}, \mathbf{Z}) = Q(\mathbf{X}, \mathbf{w}, \mathbf{Z}) = Q(\mathbf{X}, \mathbf{Y}, \mathbf{w}) = 0, \tag{15.4}$$

where $\mathbf{w}$ *is a vector from kernel of quadratic form $\hat{R}$ given by symmetric part of Ricci tensor and* $\mathbf{X}$*,* $\mathbf{Y}$*,* $\mathbf{Z}$ *are arbitrary three vectors in V.*

Proof. Relying on the results of section 8, we consider linear operator $\mathbf{F} = \mathbf{w} \otimes \alpha$, where α is an arbitrary linear functional in V. Substituting $\mathbf{F}$ into (15.3) we get

$$\alpha(\mathbf{X})\, Q(\mathbf{w}, \mathbf{Y}, \mathbf{Z}) + \alpha(\mathbf{Y})\, Q(\mathbf{X}, \mathbf{w}, \mathbf{Z}) + \alpha(\mathbf{Z})\, Q(\mathbf{X}, \mathbf{Y}, \mathbf{w}) = 0. \tag{15.5}$$

In the space V of dimension $n \geqslant 3$ for any two vectors $\mathbf{Y}$ and $\mathbf{Z}$ one can find third vector $\mathbf{X}$ and a linear functional α such that

$$\alpha(\mathbf{X}) = 1, \qquad \alpha(\mathbf{Y}) = 0, \qquad \alpha(\mathbf{Z}) = 0.$$

Then from (15.5) we get $Q(\mathbf{w}, \mathbf{Y}, \mathbf{Z}) = 0$, thus proving first relationship (15.4). Other two relationships are proved in a similar way. □

Lemma 15.2. *Proposition of lemma 15.1 remains true for the dimension $n = 2$.*

Proof. For $n = 2$ from inequalities $1 \leqslant m \leqslant n - 1$ we get $m = 1$. Due to lemma 9.2 skew-symmetric part of Ricci tensor is zero: $\tilde{\mathbf{R}} = 0$. Therefore tensor $\mathbf{Q}$ is symmetric respective to last pair of indices. For trilinear form (15.2) we obtain

$$Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = Q(\mathbf{Y}, \mathbf{X}, \mathbf{Z}). \tag{15.6}$$

Define a form $\phi(\mathbf{X}, \mathbf{Z}) = Q(\mathbf{X}, \mathbf{X}, \mathbf{Z})$, which is linear respective to $\mathbf{Z}$ and quadratic respective to $\mathbf{X}$. For this form from (15.5) we derive

$$2\, \alpha(\mathbf{X})\, Q(\mathbf{w}, \mathbf{X}, \mathbf{Z}) + \alpha(\mathbf{Z})\, \phi(\mathbf{X}, \mathbf{w}) = 0. \tag{15.7}$$

For any vector $\mathbf{X}$ in two-dimensional space one can find another vector $\mathbf{Z}$ and linear functional α such that $\alpha(\mathbf{X}) = 0$ and $\alpha(\mathbf{Z}) = 1$. Due to (15.7) this yields $\phi(\mathbf{X}, \mathbf{w}) = 0$. Form Q can be recovered by form ϕ due to its symmetry (see formula (15.6) above). We have an obvious relationship

$$Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \frac{\phi(\mathbf{X} + \mathbf{Y}, \mathbf{Z}) - \phi(\mathbf{X} - \mathbf{Y}, \mathbf{Z})}{4}. \tag{15.8}$$

Substituting $\mathbf{Z} = w$ into the formula (15.8) and taking into account $\phi(\mathbf{X}, \mathbf{w}) = 0$, we get one of the required relationships $Q(\mathbf{X}, \mathbf{Y}, \mathbf{w}) = 0$.

In order to prove rest two relationships in (15.4) we take into account that $Q(\mathbf{X}, \mathbf{Y}, \mathbf{w}) = 0$. This brings (15.5) to the following form:

$$\alpha(\mathbf{X})\, Q(\mathbf{w}, \mathbf{Y}, \mathbf{Z}) + \alpha(\mathbf{Y})\, Q(\mathbf{X}, \mathbf{w}, \mathbf{Z}) = 0. \tag{15.9}$$

For any vector $\mathbf{X}$ in two-dimensional space one can find another vector $\mathbf{Y}$ and linear functional α such that $\alpha(\mathbf{X}) = 0$ and $\alpha(\mathbf{Y}) = 1$. Therefore from (15.9) we get $Q(\mathbf{X}, \mathbf{w}, \mathbf{Z}) = 0$. The last relationship $Q(\mathbf{w}, \mathbf{Y}, \mathbf{Z}) = 0$ follows from previous one due to (15.6). Lemma 15.2 is proved. $\square$

Lemma 15.3. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1). *Under these assumptions if $n \geqslant 3$, then $\mathbf{Q} = \nabla\mathbf{R} = 0$.*

Proof. Due to lemmas 15.1 and 15.2 trilinear form Q from (15.2) is zero in the kernel $W = \operatorname{Ker}\hat{R}$ of bilinear form $\hat{R}$ from (8.3). This circumstance makes possible to defines an induced trilinear form Q in factorspace V/W where induces form $\hat{R}$ is nondegenerate. We define this form by the relationship

$$Q(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}) = Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}), \tag{15.10}$$

where $\hat{\mathbf{X}} = \operatorname{Cl}_W(\mathbf{X})$, $\hat{\mathbf{Y}} = \operatorname{Cl}_W(\mathbf{Y})$, and $\hat{\mathbf{Z}} = \operatorname{Cl}_W(\mathbf{Z})$ are cosets of three vectors $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ respective to subspace W. According to lemma 9.2 for $m \geqslant 3$ skew-symmetric part of Ricci tensor is zero. Therefore the relationship (15.6) holds and hence form (15.10) in factorspace is symmetric:

$$Q(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}) = Q(\hat{\mathbf{Y}}, \hat{\mathbf{X}}, \hat{\mathbf{Z}}). \tag{15.11}$$

By applying the complexification procedure to the factorspace V/W, if necessary, the matrix of nondegenerate symmetric bilinear form $\hat{R}$ from (8.7) can be brought to the unit matrix at the expense of proper choice of base $\mathbf{e}_1, \ldots, \mathbf{e}_m$ in factorspace V/W (see relationships (9.7)). Now consider an operator $\hat{\mathbf{F}} \in \operatorname{End}(V/W)$ defined by (9.8). For this operator we have the relationship $\hat{R}(\hat{\mathbf{F}}\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = -\hat{R}(\hat{\mathbf{X}}, \hat{\mathbf{F}}\hat{\mathbf{Y}})$, which follows from (9.7) and (9.8). Due to (9.1) operator $\hat{\mathbf{F}}$ should satisfy the equality

$$Q(\hat{\mathbf{F}}\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}) + Q(\hat{\mathbf{X}}, \hat{\mathbf{F}}\hat{\mathbf{Y}}, \hat{\mathbf{Z}}) + Q(\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{F}}\hat{\mathbf{Z}}) = 0, \tag{15.12}$$

which is obtained from (15.3) by factorization respective to subspace $W = \operatorname{Ker}\hat{R}$.

Let's substitute $\mathbf{e}_k$, $\mathbf{e}_q$, and $\mathbf{e}_i$ for $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$, and $\hat{\mathbf{Z}}$ into (15.12) in various combinations. Dimension $\dim(V/W) = m \geqslant 3$ is enough to choose indices i, k, and q to be mutually distinct. If we take into account (9.8) and substitute $\hat{\mathbf{X}} = \mathbf{e}_k$, $\hat{\mathbf{Y}} = \mathbf{e}_k$, $\hat{\mathbf{Z}} = \mathbf{e}_i$ into (15.12), we get $Q(\mathbf{e}_q, \mathbf{e}_k, \mathbf{e}_i) = -Q(\mathbf{e}_k, \mathbf{e}_q, \mathbf{e}_i)$. From the formula (15.11) in turn we get $Q(\mathbf{e}_q, \mathbf{e}_k, \mathbf{e}_i) = Q(\mathbf{e}_k, \mathbf{e}_q, \mathbf{e}_i)$. Therefore

$$Q_{qki} = Q(\mathbf{e}_q, \mathbf{e}_k, \mathbf{e}_i) = 0, \ \text{ for } \ q \neq k \neq i \neq q. \tag{15.13}$$

Now let's make other three substitutions into the formula (15.12). Their results are expressed by the following implications:

$$\begin{aligned}
&\hat{\mathbf{X}} = \mathbf{e}_i,\ \hat{\mathbf{Y}} = \mathbf{e}_i,\ \hat{\mathbf{Z}} = \mathbf{e}_k \ \text{ implies } \ Q(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_q) = 0,\\
&\hat{\mathbf{X}} = \mathbf{e}_i,\ \hat{\mathbf{Y}} = \mathbf{e}_k,\ \hat{\mathbf{Z}} = \mathbf{e}_i \ \text{ implies } \ Q(\mathbf{e}_i, \mathbf{e}_q, \mathbf{e}_i) = 0,
\end{aligned}$$

$$\hat{\mathbf{X}}=\mathbf{e}_k,\ \hat{\mathbf{Y}}=\mathbf{e}_i,\ \hat{\mathbf{Z}}=\mathbf{e}_i \ \text{ implies } \ Q(\mathbf{e}_q,\mathbf{e}_i,\mathbf{e}_i)=0.$$

Let's combine all these results into one formula

$$Q(\mathbf{e}_i,\mathbf{e}_i,\mathbf{e}_q)=Q(\mathbf{e}_i,\mathbf{e}_q,\mathbf{e}_i)=Q(\mathbf{e}_q,\mathbf{e}_i,\mathbf{e}_i)=0 \ \text{ for } \ i\neq q. \tag{15.14}$$

And finally, let's substitute $\hat{\mathbf{X}}=\mathbf{e}_k$, $\hat{\mathbf{Y}}=\mathbf{e}_q$, $\hat{\mathbf{Z}}=\mathbf{e}_q$ into (15.12). Here we get $Q(\mathbf{e}_q,\mathbf{e}_q,\mathbf{e}_q)-Q(\mathbf{e}_k,\mathbf{e}_k,\mathbf{e}_q)-Q(\mathbf{e}_k,\mathbf{e}_q,\mathbf{e}_k)=0$. If we take into account (15.14), we can bring this equality to the following form

$$Q(\mathbf{e}_q,\mathbf{e}_q,\mathbf{e}_q)=0. \tag{15.15}$$

Now summarizing (15.13), (15.14), and (15.15) we see that all components of trilinear form (15.10) in the base $\mathbf{e}_1,\ldots,\mathbf{e}_m$ of factorspace V/W are zero. This implies that Q iz zero in V/W, and hence, due to (15.10) Q iz zero in V. Therefore $\nabla\mathbf{R}=\mathbf{Q}=0$. Lemma is proved. $\square$

Lemma 15.4. *Proposition of lemma 15.3 remains true for the case $m=2$.*

Proof. For $m=2$ we are restricted by two non-coinciding values of indices $q\neq k$ for numerating base vectors in V/W. As above we suppose the relationships (9.7) to be fulfilled for the base $\mathbf{e}_1$, $\mathbf{e}_2$ in V/W and consider an operator $\hat{\mathbf{F}}\in\operatorname{End}(V,W)$ defined by it action on the base vectors:

$$\hat{\mathbf{F}}(\mathbf{e}_i)=\begin{cases}\ \ \mathbf{e}_q & \text{for } \ i=k\neq q,\\ -\mathbf{e}_k & \text{for } \ i=q\neq k.\end{cases} \tag{15.16}$$

Relying on (15.16) we make the following substitutions into the formula (15.12):

$$\hat{\mathbf{X}}=\mathbf{e}_q,\ \hat{\mathbf{Y}}=\mathbf{e}_k,\ \hat{\mathbf{Z}}=\mathbf{e}_q \ \text{ implies } \ Q(\mathbf{e}_q,\mathbf{e}_q,\mathbf{e}_q)=Q(\mathbf{e}_k,\mathbf{e}_k,\mathbf{e}_q)+Q(\mathbf{e}_q,\mathbf{e}_k,\mathbf{e}_k),$$

$$\hat{\mathbf{X}}=\mathbf{e}_k,\ \hat{\mathbf{Y}}=\mathbf{e}_q,\ \hat{\mathbf{Z}}=\mathbf{e}_q \ \text{ implies } \ Q(\mathbf{e}_q,\mathbf{e}_q,\mathbf{e}_q)=Q(\mathbf{e}_k,\mathbf{e}_k,\mathbf{e}_q)+Q(\mathbf{e}_k,\mathbf{e}_q,\mathbf{e}_k),$$

$$\hat{\mathbf{X}}=\mathbf{e}_q,\ \hat{\mathbf{Y}}=\mathbf{e}_q,\ \hat{\mathbf{Z}}=\mathbf{e}_k \ \text{ implies } \ Q(\mathbf{e}_q,\mathbf{e}_q,\mathbf{e}_q)=Q(\mathbf{e}_k,\mathbf{e}_q,\mathbf{e}_k)+Q(\mathbf{e}_q,\mathbf{e}_k,\mathbf{e}_k).$$

Let's subtract first of the above relationships from the second one and from the third one. The result can be written as

$$Q(\mathbf{e}_q,\mathbf{e}_k,\mathbf{e}_k)=Q(\mathbf{e}_k,\mathbf{e}_q,\mathbf{e}_k)=Q(\mathbf{e}_k,\mathbf{e}_k,\mathbf{e}_q). \tag{15.17}$$

To the contrary, if we add all of them and take into account (15.17), we get

$$Q(\mathbf{e}_q,\mathbf{e}_q,\mathbf{e}_q)=2\,Q(\mathbf{e}_k,\mathbf{e}_q,\mathbf{e}_k). \tag{15.18}$$

Now let's substitute $\hat{\mathbf{X}}=\mathbf{e}_k$, $\hat{\mathbf{Y}}=\mathbf{e}_k$, and $\hat{\mathbf{Z}}=\mathbf{e}_k$ into (15.12). This yields

$$Q(\mathbf{e}_q,\mathbf{e}_k,\mathbf{e}_k)+Q(\mathbf{e}_k,\mathbf{e}_q,\mathbf{e}_k)+Q(\mathbf{e}_k,\mathbf{e}_k,\mathbf{e}_q)=0. \tag{15.19}$$

From (15.17), (15.18), and (15.19) we see that all components of trilinear form Q in the base $\mathbf{e}_1$, $\mathbf{e}_2$ of factorspace V/W are zero:

$$Q(\mathbf{e}_q, \mathbf{e}_k, \mathbf{e}_k) = Q(\mathbf{e}_k, \mathbf{e}_q, \mathbf{e}_k) = Q(\mathbf{e}_k, \mathbf{e}_k, \mathbf{e}_q) = Q(\mathbf{e}_q, \mathbf{e}_q, \mathbf{e}_q) = 0 \quad \text{for} \quad q \neq k.$$

Then from (15.10) we get $Q = 0$, thus completing the the proof of lemma 15.4. $\square$

For $m = 1$ tensor $\mathbf{Q} = \nabla \mathbf{R}$ shouldn't vanish. However, in this case factorspace V/W is one-dimensional. Trilinear form (15.10) in V/W has only one component $Q_{111} = Q(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_1)$. Therefore trilinear form Q in the initial space V is determined by some linear form u up to a scalar factor:

$$Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \sim u(\mathbf{X})\, u(\mathbf{Y})\, u(\mathbf{Z}). \tag{15.20}$$

For tensor field $\mathbf{Q} = \nabla \mathbf{R}$ formula (15.20) is written as $\mathbf{Q} = \nabla \mathbf{R} \sim \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$. This means that trilinear form (15.20) is completely symmetric. For covariant differential of Ricci tensor this yields the following symmetry:

$$\nabla_i R_{jk} = \nabla_j R_{ik}. \tag{15.21}$$

Ricci tensor itself for $m = 1$ is also symmetric: $\mathbf{R} = \hat{\mathbf{R}}$ and $S = 2\,\tilde{\mathbf{R}} = 0$, as stated in lemma 9.2. For bilinear form $\hat{R}$ we have the relationship

$$\hat{R}(\mathbf{X}, \mathbf{Y}) \sim u(\mathbf{X})\, u(\mathbf{Y}), \tag{15.22}$$

similar to (15.20). Due to (15.22) linear form $u(\mathbf{X})$ should vanish when applied to the vector $\mathbf{X}$ in the kernel $W = \operatorname{Ker} \hat{R}$.

16. Two-dimensional projectively-euclidean spaces.

For $n = 2$ we have only one case of intermediate degeneration with $m = 1$. Suppose that system of equations (1.1) belongs to this case and suppose that it has an algebra of point symmetries of maximal dimension (9.1). Ricci tensor for the connection Γ defined by this system of equations is symmetric ($\mathbf{R} = \hat{\mathbf{R}}$, $\tilde{\mathbf{R}} = 0$). Curvature tensor has the form (12.1), which is same as (10.15) if we take into account $n = 2$ and $\tilde{\mathbf{R}} = 0$. Therefore we can transform (12.1) to (14.1) if we denote $\beta_{ij} = \hat{R}_{ij} = R_{ij}$. The equations (14.2) for the field $\boldsymbol{\beta}$ in two-dimensional case are not immediate consequence of (12.1) (Bianchi-Padov identity for $n = 2$ is trivial). However, they follow from (9.1) due to lemma 15.2 which leads to (15.21). Since the equations (14.2) are fulfilled, we can repeat arguments from section 14 for two-dimensional case $n = 2$. This results in the following theorem similar to theorem 14.1.

Theorem 16.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimen-*

sion (9.1). *Under these assumptions if* $n = 2$, *then there is a point transformation* (1.4) *bringing these equations to the form*

$$\frac{\partial y^i}{\partial t} = a\,\frac{\partial^2 y^i}{\partial x^2} - 2\,a\left(\sum_{r=1}^{2} u_r\,\frac{\partial y^r}{\partial x}\right)\frac{\partial y^i}{\partial x},\quad i = 1,\,2,$$

where $a = \mathrm{const}$ *and* $u_r = u_r(y^1, y^2)$ *are components of some covector field* $\mathbf{u}$.

17. Separation of variables.

Due to results of section 16 we can make further consideration common for all cases of intermediate degeneration $1 \leqslant m \leqslant n-1$ and for all dimensions $n \geqslant 2$. In order to construct projectively-euclidean coordinates in section 14 we chose some solution of the system of Pfaff equations (14.3) defining some covector field $\mathbf{u}$ on the manifold M. Now we shall make the choice of field $\mathbf{u}$ more specific. In order to do it we complete the equations (14.3) with the equations for some vector field $\boldsymbol{\xi}$:

$$\begin{cases} \nabla_i u_k = \beta_{ik} + u_i\,u_k,\\ \nabla_i \xi^k = -u_i\,\xi^k. \end{cases} \tag{17.1}$$

Here $i, k = 1, \ldots, n$. The equations (17.1) form complete system of Pfaff equations. However, in contrast to (14.3) this system of Pfaff equations is not completely compatible. In order to make it compatible we impose the following restrictions for the variables $u_1, \ldots, u_n$ and $\xi^1, \ldots, \xi^n$:

$$\sum_{r=1}^{n} \hat{\beta}_{ir}\,\xi^r = 0, \qquad\qquad \sum_{r=1}^{n} u_r\,\xi^r = 0 \tag{17.2}$$

(brief introduction into the theory of Pfaff equations with restrictions can be found in [23], appendix B). Let's check the compatibility of (17.1) due to restrictions (17.2). For to do it we calculate the commutator of covariant derivatives according to the equations (17.1). This is implemented by the following two equations:

$$\begin{aligned} &-\sum_{s=1}^{n} R^s_{kij}\,u_s = [\nabla_i, \nabla_j]u_k = \nabla_i(\beta_{jk} + u_j\,u_k) - \nabla_j(\beta_{ik} + u_i\,u_k),\\ &\sum_{s=1}^{n} R^k_{sij}\,\xi^s = [\nabla_i, \nabla_j]\xi^k = \nabla_i(-u_j\,\xi^k) - \nabla_j(u_i\,\xi^k). \end{aligned} \tag{17.3}$$

First equation (17.3) is an identity due to (14.1) and (14.2), here we need no restrictions. Second one is reduced to

$$\sum_{s=1}^{n} \beta_{js}\,\xi^s\,\delta^k_i - \sum_{s=1}^{n} \beta_{is}\,\xi^s\,\delta^k_j = 0. \tag{17.4}$$

The equality (17.4) holds due to the first equation of restrictions (17.2). Indeed, (17.2) means that vector $\boldsymbol{\xi}$ belongs to the kernel of quadratic form $\hat{R}$. By lemma 9.1

it follows that the kernel of skew-symmetric part of the form β contains the kernel of its symmetric part: $\operatorname{Ker}\tilde{\beta} = \operatorname{Ker} S = \operatorname{Ker}\tilde{R} \supset \operatorname{Ker}\hat{R}$.

Now let's prove that the restrictions (17.2) are in agreement with the equations (17.1). In order to do it let's calculate covariant derivatives of left hand sides of (17.2) and equate them to zero:

$$0 = \sum_{r=1}^{n} \nabla_i(\hat{\beta}_{jr}\,\xi^r) = \sum_{r=1}^{n} \nabla_i\hat{\beta}_{jr}\,\xi^r - \sum_{r=1}^{n} u_i\,\hat{\beta}_{jr}\,\xi^r, \tag{17.5}$$

$$0 = \sum_{r=1}^{n} \nabla_i(u_r\,\xi^r) = \sum_{r=1}^{n} \beta_{ir}\,\xi^r. \tag{17.6}$$

The equality (17.6) is fulfilled due to the first equation of restrictions (17.2). The equality (17.5) also is provided by this equation and lemmas 15.1 and 15.2.

Compatibility of the Pfaff equations (17.1) restricted by the equations (17.2) means that for these equations the Cauchy problem

$$u_k\Bigr|_{p=p_0} = u_k(0), \qquad\qquad \xi^k\Bigr|_{p=p_0} = \xi^k(0) \tag{17.7}$$

is uniquely solvable in some neighborhood of the point p_0 for arbitrary initial values satisfying the restrictions (17.2). Solution of this Cauchy problem satisfies the equations of restrictions (17.2) in whole neighborhood where it is defined.

Let's choose one set of initial values for the components of covector $\mathbf{u}$ and $n-m$ sets of initial values for the components of vector $\boldsymbol{\xi}$. Vectors $\boldsymbol{\xi}_1(0), \ldots, \boldsymbol{\xi}_{n-m}(0)$ should be chosen so that they are linearly independent and all belong to the kernel of the form $\hat{R}$. This provides first condition (17.2). Covector $\mathbf{u}(0)$ should be chosen such that it vanishes being contracted with the vectors of the subspace $\operatorname{Ker}\hat{R}$. This provides second condition (17.2). Pfaff equation (17.1) respective to $\mathbf{u}$ do not depend on components of $\boldsymbol{\xi}$. Therefore by solving $n-m$ Cauchy problems we obtain $n-m$ vector fields $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$ and only one covector field $\mathbf{u}$. Vector fields $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$ constitute moving frame forming the base of subspace $\operatorname{Ker}\hat{R}$ at each point on M. Covector field $\mathbf{u}$ vanishes being contracted with vectors of frame $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$. We express this circumstance as

$$u(\boldsymbol{\xi}_1) = \ldots = u(\boldsymbol{\xi}_{n-m}) = 0. \tag{17.8}$$

Using (17.1) and the relationships (17.8) we can calculate covariant derivatives of the fields $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$, and $\mathbf{u}$ along the vectors of the frame $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$:

$$\nabla_{\boldsymbol{\xi}_i}\boldsymbol{\xi}_j = 0, \qquad\qquad \nabla_{\boldsymbol{\xi}_i}\mathbf{u} = 0. \tag{17.9}$$

Then from (17.9) we derive the commutators

$$[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j] = \nabla_{\boldsymbol{\xi}_i}\boldsymbol{\xi}_j - \nabla_{\boldsymbol{\xi}_j}\boldsymbol{\xi}_i = 0. \tag{17.10}$$

Following theorem is an immediate consequence of the relationships (17.10).

Theorem 17.1. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions kernels of bilinear forms $\hat{R}$ defined by symmetric part of Ricci tensor $\hat{\mathbf{R}}$ form involutive $(n-m)$-dimensional distribution on M, which is integrable by Frobenius theorem.*

Retaining field $\mathbf{u}$ defined by (17.1) and (17.7) as described above let's consider another system of Pfaff equations for the components of vector field $\mathbf{X}$:

$$\nabla_i X^k = -u_i\, X^k - \sum_{r=1}^{n} u_r\, X^r\, \delta^k_i, \quad i, k = 1, \ldots, n. \tag{17.11}$$

Pfaff equations (17.11) are completely compatible without imposing any restrictions. For the equations (17.11) the Cauchy problem similar to (17.7) is solvable for arbitrary initial values in it:

$$X^k\Big|_{p=p_0} = X^k(0). \tag{17.12}$$

Let's choose m sets of initial values in (17.12) such that cosets of corresponding vectors $\mathbf{X}_1(0), \ldots, \mathbf{X}_m(0)$ respective to subspace $W = \operatorname{Ker}\hat{R}$ are linear independent in factorspace V/W. By solving Cauchy problems (17.12) for (17.11) with these sets of initial values we obtain m vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_m$ that in aggregate with $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$ form n-dimensional moving frame on M. Denote by $u(\mathbf{X}_1), \ldots, u(\mathbf{X}_m)$ the results of contraction of covector field $\mathbf{u}$ with vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_m$. Then, by direct calculations, from (17.11) we get

$$\nabla_{\mathbf{X}_i}\mathbf{X}_j = -u(\mathbf{X}_i)\mathbf{X}_j - u(\mathbf{X}_j)\mathbf{X}_i, \tag{17.13}$$

Analogously from (17.1) and (17.11) we derive

$$\nabla_{\mathbf{X}_i}\boldsymbol{\xi}_j = -u(\mathbf{X}_i)\boldsymbol{\xi}_j, \qquad \nabla_{\boldsymbol{\xi}_i}\mathbf{X}_j = -u(\mathbf{X}_j)\boldsymbol{\xi}_i. \tag{17.14}$$

By (17.14) we find cross commutators of $\mathbf{X}_1, \ldots, \mathbf{X}_m$ and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$:

$$[\boldsymbol{\xi}_i, \mathbf{X}_j] = \nabla_{\boldsymbol{\xi}_i}\mathbf{X}_j - \nabla_{\mathbf{X}_j}\boldsymbol{\xi}_i = 0. \tag{17.15}$$

Moreover, vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_m$ are commutating with each other:

$$[\mathbf{X}_i, \mathbf{X}_j] = \nabla_{\mathbf{X}_i}\mathbf{X}_j - \nabla_{\mathbf{X}_j}\mathbf{X}_i = 0. \tag{17.16}$$

This follows from (17.13). Further we shall calculate derivatives of the functions $u(\mathbf{X}_1), \ldots, u(\mathbf{X}_m)$ along vector fields $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$:

$$\nabla_{\boldsymbol{\xi}_i}(u(\mathbf{X}_j)) = C(\nabla_{\boldsymbol{\xi}_i}\mathbf{u} \otimes \mathbf{X}_j + \mathbf{u} \otimes \nabla_{\boldsymbol{\xi}_i}\mathbf{X}_j) = 0. \tag{17.17}$$

From (17.10), (17.15), and (17.16) we conclude that moving frame $\mathbf{X}_1, \ldots, \mathbf{X}_m$, $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$ consists of n mutually commutating vector fields. Therefore one can find local coordinates $y^1, \ldots, y^n$ whose frame of coordinate tangent fields coincides with $\mathbf{X}_1, \ldots, \mathbf{X}_m, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-m}$. Formulas (17.9), (17.13), and (17.14) determine connection components in these local coordinates:

$$\Gamma^k_{rs} = -(u_r\,\delta^k_s + u_s\,\delta^k_r). \tag{17.18}$$

Here $u_k = 0$ for $k > m$ and $u_k = u_k(y^1, \ldots, y^m)$ for $1 \leqslant k \leqslant m$. This follows from (17.8) and (17.17). Comparing (17.18) and (14.7) we state the theorem strengthening the result of theorem 14.1.

Theorem 17.2. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1)*. Under these assumptions there is a point transformation* (1.4) *bringing these equations to the form*

$$\frac{\partial y^i}{\partial t} = a\,\frac{\partial^2 y^i}{\partial x^2} - 2\,a\left(\sum_{r=1}^{m} u_r\,\frac{\partial y^r}{\partial x}\right)\frac{\partial y^i}{\partial x}, \quad i = 1, \ldots, n, \tag{17.19}$$

where $a = \mathrm{const}$ *and* $u_r = u_r(y^1, \ldots, y^m)$ *are components of some covector field* $\mathbf{u}$.

Let $m = 2$. From lemmas 15.3 and 15.4 we know that $\nabla_q \beta_{ij} = 0$. Therefore $\nabla_k \nabla_q \beta_{ij} = 0$. Let's write the identity

$$-\sum_{s=1}^{n} R^s_{ikq}\,\beta_{sj} - \sum_{s=1}^{n} R^s_{jkq}\,\beta_{is} = [\nabla_k,\, \nabla_q]\,\beta_{ij} = 0$$

and substitute the curvature tensor components (14.1) into this identity:

$$4\,\tilde\beta_{kq}\,\beta_{ij} + \beta_{ki}\,\beta_{qj} - \beta_{qi}\,\beta_{kj} + \beta_{kj}\,\beta_{iq} - \beta_{qj}\,\beta_{ik} = 0.$$

Hence $2\,\tilde\beta_{kq}\,\beta_{ij} + \tilde\beta_{ki}\,\beta_{qj} - \tilde\beta_{qi}\,\beta_{kj} = 0$. This relationship is skew-symmetric respective to k and q. We symmetrize it respective to i and j:

$$4\,\tilde\beta_{kq}\,\hat\beta_{ij} + \tilde\beta_{ki}\,\hat\beta_{qj} + \tilde\beta_{kj}\,\hat\beta_{qi} - \tilde\beta_{qi}\,\hat\beta_{kj} - \tilde\beta_{ki}\,\hat\beta_{qj} = 0. \tag{17.20}$$

In the above projectively-euclidean coordinates $y^1, \ldots, y^n$ the only nonzero components of tensor $\tilde{\boldsymbol{\beta}}$ for $m = 2$ are $\tilde\beta_{12}$ and $\tilde\beta_{21} = -\tilde\beta_{12}$. Tensor $\hat{\boldsymbol{\beta}}$ in these coordinates can have only four nonzero components, they are $\hat\beta_{11}$, $\hat\beta_{22}$, $\hat\beta_{12}$, and $\hat\beta_{21} = \hat\beta_{12}$. For $k = 1$ and $q = 2$ from (17.20) we get

$$\tilde\beta_{12}\,\hat\beta_{11} = 0, \qquad \tilde\beta_{12}\,\hat\beta_{12} = 0, \qquad \tilde\beta_{12}\,\hat\beta_{22} = 0. \tag{17.21}$$

Following three components $\hat\beta_{11}$, $\hat\beta_{12}$, and $\hat\beta_{22}$ cannot vanish simultaneously. Therefore from (17.21) we get $\tilde\beta_{12} = -\tilde\beta_{21} = 0$, which leads to $(n+1)\,\tilde{\boldsymbol{\beta}} = -\tilde{\mathbf{R}} = 0$.

Lemma 17.1. *Proposition of lemma 9.2 remains true for $m = 2$.*

Let's alternate (14.3) respective to i and j and let's take into account symmetry of connection components. Due to $\beta_{ij} = \beta_{ji}$ we obtain

$$\frac{\partial u_j}{\partial y^i} - \frac{\partial u_i}{\partial y^j} = 0. \tag{17.22}$$

The relationships (17.22) are exactly the compatibility conditions for the following system of Pfaff equations for the scalar field ψ:

$$\frac{\partial \psi}{\partial y^i} = u_i(y^1, \ldots, y^m), \quad i = 1, \ldots, m. \tag{17.23}$$

Due to the integrability of equations (17.23) we can state a theorem which makes (17.19) more specific.

Theorem 17.3. *Let* (1.1) *be a system of equations belonging to m-th case of intermediate degeneration such that its algebra of point symmetries has maximal dimension* (9.1). *Under these assumptions there is a point transformation* (1.4) *bringing these equations to the form*

$$\frac{\partial y^i}{\partial t} = a\,\frac{\partial^2 y^i}{\partial x^2} - 2\,a\left(\sum_{r=1}^{m}\frac{\partial \psi}{\partial y^r}\,\frac{\partial y^r}{\partial x}\right)\frac{\partial y^i}{\partial x}, \quad i = 1, \ldots, n, \tag{17.24}$$

where $a = \mathrm{const}$ and $\psi = \psi(y^1, \ldots, y^m)$ is some scalar field.

18. Case of general position.

Case of general position is distinguished by the condition $m = n \geqslant 2$, which means that the kernel of bilinear form defined by symmetric part of Ricci tensor is zero: $\operatorname{Ker}\hat{R} = 0$. The estimate (8.9) for this case yields

$$\dim(\mathcal{G}) \leqslant \frac{n(n+1)}{2}. \tag{18.1}$$

Our further aim is to describe systems of equations (1.1) for which the upper bound in the estimate (18.1) is reached:

$$\dim(\mathcal{G}) = \frac{n(n+1)}{2}. \tag{18.2}$$

Let's start with study of the fields $\mathbf{S}$ and $\tilde{\mathbf{R}}$ from (4.11) and (4.12). For these fields we have a lemma similar to lemma 9.2.

Lemma 18.1. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *Under these assumptions $\mathbf{S} = 2\,\tilde{\mathbf{R}} = 0$.*

Proof of this lemma is almost literally the same as the proof of lemma 9.2. The difference is only that $\operatorname{Ker}\hat{R} = 0$. Therefore we need not use factorspace and factoroperators in proving lemma 18.1.

Lemma 18.2. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *Under these assumptions if* $n \geqslant 3$*, then* $\mathbf{Q} = \nabla\mathbf{R} = 0$.

Lemma 18.3. *Proposition of lemma 18.2 remains true for the dimension* $n = 2$.

Proof of the lemma 18.2 is based on lemma 18.1. It is the same as the proof of lemma 15.3 with the only difference $W = \operatorname{Ker}\hat{R} = 0$. Therefore we can do without factorspace V/W. Proof of lemma 18.3 is analogous to that of the lemma 15.4.

In the case of general position tensor $\hat{\mathbf{R}}$ is nondegenerate. Therefore it defines pseudoriemannian metric $\mathbf{g} = \hat{\mathbf{R}}$ on M. From lemmas 18.2 and 18.3 we find that this metric is in concordance with the connection Γ:

$$\nabla_k g_{ij} = \nabla_k \hat{R}_{ij} = \frac{\nabla_k R_{ij} + \nabla_k R_{ji}}{2} = 0. \tag{18.3}$$

Components of symmetric connection Γ concordant with metric tensor $\mathbf{g}$ are calculated by well-known formula

$$\Gamma^k_{ij} = \sum_{i=1}^{n} \frac{g^{ks}}{2} \left(\frac{\partial g_{sj}}{\partial y^i} + \frac{\partial g_{is}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^s} \right). \tag{18.4}$$

Formula (18.4) for Γ^k_{ij} follows from (18.3) (details see in [6], [9], or [17]). Using metric tensor $\mathbf{g}$ one can lower upper index in curvature tensor. This leads to the tensor field of type $(0,4)$ with components

$$R_{kqij} = \sum_{i=1}^{n} g_{ks}\, R^s_{qij}. \tag{18.5}$$

Theorem 18.1. *Components of curvature tensor* (18.5) *for metric connection* (18.4) *possess two symmetry properties*

$$R_{kqij} = -R_{qkij}, \qquad\qquad R_{kqij} = R_{ijkq}, \tag{18.6}$$

in addition to the properties $R^k_{qij} = -R^k_{qji}$ *and* $R^k_{qij} + R^k_{ijq} + R^k_{jqi} = 0$*, which are available for curvature tensor of any symmetric connection.*

Proposition of theorem 18.1 is well-known fact. Its proof can be found in [6] or in [17]. As an immediate consequence of (18.6) we obtain symmetry of Ricci tensor:

$$R_{qj} = \sum_{i=1}^{n}\sum_{k=1}^{n} g^{ki}\, R_{kqij} = \sum_{i=1}^{n}\sum_{k=1}^{n} g^{ik}\, R_{ijkq} = R_{jq}.$$

This relationship can be used to strengthen the proposition of lemma 18.1 by extending it for the case of two-dimensional manifolds.

Lemma 18.4. *Proposition of lemma 18.1 remains true for the dimension* $n = 2$.

19. Structure of curvature tensor for $n \geqslant 3$.

Let's fix a point p_0 on the manifold M and consider a field of point symmetry $\boldsymbol{\eta}$ of the system of equations (1.1) vanishing at the point p_0. Such field generates linear operator (5.7) in tangent space $T_{p_0}(M)$. Denote it by $\mathbf{F}$. According to the theorem 4.1 we have $L_{\boldsymbol{\eta}}(\mathbf{R}) = 0$. This equation written at the point p_0 takes the form (10.2). Here we suppose that system of equations (1.1) belongs to the case of general position and the dimension of its symmetry algebra is given by (18.2). Due to (18.2) any operator $\mathbf{F}$ satisfying (8.3) corresponds to some symmetry field $\boldsymbol{\eta} \in \mathcal{G}(p_0)$. In the case of general position bilinear form $g = \hat{R}$ in (8.3) is nondegenerate. By applying the complexification procedure to the space V, if necessary, the matrix of the form g can be brought to the unit matrix at the expense of proper choice of base $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $V = T_{p_0}(M)$:

$$g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j) = \hat{R}(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases} \tag{19.1}$$

For the dimension $n \geqslant 3$ we can consider three base vectors $\mathbf{e}_k$, $\mathbf{e}_q$, $\mathbf{e}_r$ such that $k \neq q \neq r \neq k$. Let's define an operator $\mathbf{F}$ by its action on the base vectors:

$$\hat{\mathbf{F}}(\mathbf{e}_i) = \begin{cases} \mathbf{e}_q & \text{for} \quad i = k, \\ -\mathbf{e}_k & \text{for} \quad i = q, \\ 0 & \text{for} \quad i \neq k \quad \text{and} \quad i \neq q. \end{cases} \tag{19.2}$$

One can easily check that for operator $\mathbf{F}$ the relationships (8.3) are fulfilled. Let's substitute $\mathbf{F}$ into the formula (10.2) and take $\mathbf{X} = \mathbf{e}_k$, $\mathbf{Y} = \mathbf{e}_r$, $\mathbf{Z} = \mathbf{e}_k$ in it. This yields $\mathbf{R}(\mathbf{F}\mathbf{e}_k, \mathbf{e}_r)\mathbf{e}_k + \mathbf{R}(\mathbf{e}_k, \mathbf{e}_r)\mathbf{F}\mathbf{e}_k = \mathbf{F}\mathbf{R}(\mathbf{e}_k, \mathbf{e}_r)\mathbf{e}_k$. Now by (19.2) we obtain

$$\sum_{i=1}^{n} R^i_{kqr}\,\mathbf{e}_i + \sum_{i=1}^{n} R^i_{qkr}\,\mathbf{e}_i = \sum_{s=1}^{n} R^s_{kkr}\,\mathbf{F}\mathbf{e}_s = R^k_{kkr}\,\mathbf{e}_q - R^q_{kkr}\,\mathbf{e}_k.$$

This vectorial equality is reduced to the series of scalar equalities:

$$R^k_{kqr} + R^k_{qkr} = -R^q_{kkr} \quad \text{for} \quad i = k, \tag{19.3}$$

$$R^q_{kqr} + R^q_{qkr} = R^k_{kkr} \quad \text{for} \quad i = q, \tag{19.4}$$

$$R^i_{kqr} + R^i_{qkr} = 0 \quad \text{for} \quad i \neq k \quad \text{and} \quad i \neq q. \tag{19.5}$$

Due to (19.1) we make no difference between upper and lower indices for tensor components referred to the base $\mathbf{e}_1, \ldots, \mathbf{e}_n$. By means of (18.6) from (19.3), (19.4), and (19.5) we derive the following equalities:

$$R^k_{kqr} = 0, \quad R^q_{kqr} = 0, \;\text{ and }\; R^i_{kqr} = -R^i_{qkr} \quad \text{for} \quad i \neq k \quad \text{and} \quad i \neq q.$$

Note that we can unite these three equalities into one:

$$R^i_{kqr} = -R^i_{qkr} \quad \text{for} \quad k \neq q \neq r \neq k. \tag{19.6}$$

Let's take into account (19.6) and write the identity $R^i_{kqr}+R^i_{qrk}+R^i_{rkq}=0$, which follows from symmetry $\Gamma^i_{rs}=\Gamma^i_{sr}$. This results in

$$R^i_{kqr}=0 \quad\text{for}\quad q\neq k\neq r. \tag{19.7}$$

As a next step let's substitute $\mathbf{F}$ into the formula (10.2) taking $\mathbf{X}=\mathbf{e}_k$, $\mathbf{Y}=\mathbf{e}_r$, $\mathbf{Z}=\mathbf{e}_q$ in it: $\mathbf{R}(\mathbf{F}\mathbf{e}_k,\mathbf{e}_r)\mathbf{e}_q+\mathbf{R}(\mathbf{e}_k,\mathbf{e}_r)\mathbf{F}\mathbf{e}_q=\mathbf{F}\mathbf{R}(\mathbf{e}_k,\mathbf{e}_r)\mathbf{e}_q$. By (19.2) we get

$$\sum_{i=1}^{n}R^i_{qqr}\,\mathbf{e}_i-\sum_{i=1}^{n}R^i_{kkr}\,\mathbf{e}_i=\sum_{s=1}^{n}R^s_{qkr}\,\mathbf{F}\mathbf{e}_s=R^k_{qkr}\,\mathbf{e}_q-R^q_{qkr}\,\mathbf{e}_k.$$

This vectorial equality is reduced to the series of scalar equalities:

$$\begin{aligned}
&R^k_{qqr}-R^k_{kkr}=-R^q_{qkr}\quad\text{for}\quad i=k,\\
&R^q_{qqr}-R^q_{kkr}=R^k_{qkr}\quad\text{for}\quad i=q,\\
&R^i_{qqr}-R^i_{kkr}=0\quad\text{for}\quad i\neq k\quad\text{and}\quad i\neq q.
\end{aligned}$$

We choose the last equality and substitute $i=r$ in it. As a result we obtain

$$R^r_{qqr}=R^r_{kkr}\quad\text{for}\quad q\neq r\neq k. \tag{19.8}$$

Due to (19.7) we conclude that R^k_{qsr} can be nonzero only if some two of three lower indices are equal, i. e. if $q=r$ or $q=s$ (chance $s=r$ is excluded since R^k_{qsr} is skew-symmetric relative to indices s and r):

$$R^k_{qsr}=g_{qr}\,\beta^k_s(q)-g_{qs}\,\beta^k_r(q). \tag{19.9}$$

In formula (19.9) we took into account (19.1) and skew-symmetry $R^k_{qsr}=-R^k_{qrs}$. Now we are to specify the quantities $\beta^k_s(q)$ and $\beta^k_r(q)$ in (19.9).

Let $q\neq r$. In order to find $\beta^k_r(q)$ we substitute $s=q$ into the formula (19.9). Upon some easy calculations we get

$$\beta^k_r(q)=-R^k_{qqr}=R_{kqqr}=R_{qkqr}=R^q_{kqr}\quad\text{for}\quad q\neq r. \tag{19.10}$$

Here we took into account (18.6) and the property (19.1) of the base $\mathbf{e}_1,\,\ldots\,,\mathbf{e}_n$, due to which we can mix upper and lower indices. Let's transform the right hand side of (19.10). For $q\neq k\neq r$ right hand side of (19.10) is zero, one should substitute $i=q$ into (19.7) in order to see it. For $q=k\neq r$ it is zero too due to $R^q_{qqr}=R_{qqqr}$ and (18.6). Therefore

$$\beta^k_r(q)=0\quad\text{for}\quad q\neq r\neq k. \tag{19.11}$$

Suppose $q\neq k=r$. Then from (19.10) we get $\beta^r_r(q)=-R^r_{qqr}$. Formula (19.8) and the equalities $R^r_{rrr} = 0$ and $R^r_{kkr} = R_{rkkr} = -R_{krkr} = -R^k_{rkr}$ are used for further transformation of the expression for $\beta^r_r(q)$:

$$\beta^r_r(q) = -R^r_{qqr} = -\sum_{k\neq r}^{n} \frac{R^r_{kkr}}{n-1} = \sum_{k=1}^{n} \frac{R^k_{rkr}}{n-1} = \frac{R_{rr}}{n-1} = \frac{g_{rr}}{n-1} = \frac{1}{n-1}.$$

We combine this result with (19.11) and we write it as follows:

$$\beta^k_r(q) = \frac{\delta^k_r}{n-1} \quad \text{for} \quad q \neq r. \tag{19.12}$$

Formula (19.12) do not define $\beta^k_r(q)$ for $q = r$. For arbitrary values of k, q, and r we replace it by the following formula, equivalent to (19.12) for $q \neq r$:

$$\beta^k_r(q) = \frac{\delta^k_r}{n-1}(1 - g_{qr}) + \beta^k_r(r)\, g_{qr}. \tag{19.13}$$

Now substitute (19.13) into the formula (19.9). As a result we get

$$R^k_{qsr} = \frac{g_{qr}\,\delta^k_s - g_{qs}\,\delta^k_r}{n-1}. \tag{19.14}$$

It's worth to compare (19.14) with (10.15) and take into account lemma 18.1, due to which Ricci tensor $\mathbf{g} = \mathbf{R}$ is symmetric.

20. Structure of curvature tensor for $n = 2$.

For $n = 2$ Ricci tensor is symmetric as well. This follows from lemma 18.4. Formula (19.14) for $n = 2$ takes the following form:

$$R^k_{qsr} = g_{qr}\,\delta^k_s - g_{qs}\,\delta^k_r. \tag{20.1}$$

The proof or the formula (20.1) is the same as fore the formula (12.1).

Theorem 20.1. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *Under these assumptions if $n = 2$, then components of curvature tensor are expressed through components Ricci tensor* $\mathbf{g} = \mathbf{R}$ *according to the formula* (19.4).

21. Structure of tensor field $\mathbf{A}$

Theorem 21.1. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *Under these assumptions if $n \geqslant 3$, then A is a constant scalar matrix, i. e. $A^i_j = a\,\delta^i_j$ and $a =$* const.

Proof. Let's fix a point p_0 on M and consider a field of point symmetry $\boldsymbol{\eta}$ for the system of equations (1.1). Suppose that $\boldsymbol{\eta}$ vanishes at p_0. The equation $L_{\boldsymbol{\eta}}(\mathbf{A})=0$ from (2.5) for this field at the point p_0 is written as

$$\mathbf{A}\,\mathbf{F}\mathbf{X}=\mathbf{F}\,\mathbf{A}\mathbf{X}. \tag{21.1}$$

Here $\mathbf{F}$ is linear operator from (5.7) and $\mathbf{X}$ is an arbitrary vector from tangent space $V=T_{p_0}(M)$. Due to (18.2) for $\mathbf{F}$ we can take any operator satisfying the equations (8.3). In the case of general position bilinear form $g=\hat{R}$ in (8.3) is nondegenerate. By applying the complexification procedure to the space V, if necessary, the matrix of nondegenerate symmetric bilinear form $\hat{R}$ from (8.7) can be brought to the unit matrix at the expense of proper choice of base $\mathbf{e}_1,\,\ldots\,,\,\mathbf{e}_m$, i. e. there is a base such that the relationships (19.1) hold.

For the dimension $n\geqslant 3$ we can find three base vectors $\mathbf{e}_k$, $\mathbf{e}_q$, $\mathbf{e}_r$ such that $k\neq q\neq r\neq k$. Let's define the operator $\mathbf{F}$ by means of (19.2) and substitute it into the relationship (21.1) taking $\mathbf{X}=\mathbf{e}_k$. This yields

$$\sum_{i=1}^{n}A^i_q\,\mathbf{e}_i=\sum_{s=1}^{n}A^s_k\,\mathbf{F}\mathbf{e}_s=A^k_k\,\mathbf{e}_q-A^q_k\,\mathbf{e}_k.$$

The above vectorial equality reduces to the series if scalar equalities:

$$A^q_q=A^k_k,\quad A^k_q=-A^q_k,\ \text{ and }\ A^i_q=0\ \text{ for }\ i\neq k\ \text{ and }\ i\neq q. \tag{21.2}$$

First equality $A^q_q=A^k_k$ means, that all diagonal elements of matrix A are equal to each other. The last equality $A^i_q=0$ means that nondiagonal elements are zero. Hence $A^i_j=a\,\delta^i_j$ or $\mathbf{A}=a\;\mathbf{id}$, where a is a scalar field. Substituting $\mathbf{A}=a\;\mathbf{id}$ into the equation $L_{\boldsymbol{\eta}}(\mathbf{A})=0$ we get the equation (13.4), which leads to $L_{\boldsymbol{\eta}}(a)=0$. Due to (18.2) we can find n vector fields $\boldsymbol{\eta}_1,\,\ldots\,,\,\boldsymbol{\eta}_n$ of point symmetry algebra $\mathcal{G}$ which are linear independent at the point p_0. The equation $L_{\boldsymbol{\eta}}(a)=0$ holds for each of these fields. Therefore $a=\text{const}$. Theorem is proved. $\square$

On the orientable manifold of the dimension $n=2$ equipped with nondegenerate metric $\mathbf{g}$ apart from the field of identical operators $\mathbf{id}$ there exists a field of rotation by the angle 90° in metric $\mathbf{g}$. Denote it by $\mathbf{P}$. Components of tensor field $\mathbf{P}$ are defined by the following explicit formula:

$$P^i_j=\pm\sum_{s=1}^{2}\frac{d^{is}\,g_{sj}}{\sqrt{|\det g|}}. \tag{21.3}$$

Here d^{is} are components of skew-symmetric unit matrix (12.2). Sign in the formula (21.3) is defined by the orientation of the system of local coordinates. Upon complexification of the tangent space V if we choose the base where the relationships (19.1) are fulfilled, then for the matrix of the operator field $\mathbf{P}$ we obtain

$$P=\varepsilon\cdot\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}, \tag{21.4}$$

where $\varepsilon = \pm 1$ for positive or negative metric $\mathbf{g}$ and $\varepsilon = \pm i$ for indefinite metric $\mathbf{g}$.

Theorem 21.2. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *If $n = 2$, then components of the matrix A are given by the formula*

$$A^i_j = a\,\delta^i_j + b\sum_{s=1}^{2}\frac{d^{is}\,R_{sj}}{\sqrt{|\det R|}}, \tag{21.5}$$

where a and b are constants, $R_{sj} = g_{sj}$ are components of Ricci tensor for the connection Γ, and $\det R$ is the determinant of the matrix of this tensor.

Proof. For the dimension $n = 2$ we have only two base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Let's define the operator $\mathbf{F}$ by its action on base vectors:

$$\mathbf{F}(\mathbf{e}_i) = \begin{cases} \mathbf{e}_2 & \text{for}\quad i = 1,\\ -\mathbf{e}_1 & \text{for}\quad i = 2.\end{cases}$$

When we substitute $\mathbf{F}$ into (21.1) and take $\mathbf{X} = \mathbf{e}_1$, instead of (21.2) here we get $A^2_2 = A^1_1 = a$ and $A^2_1 = -A^1_2 = -b$. Therefore matrix of operator $\mathbf{A}$ has the form

$$A = \begin{Vmatrix} a & b\\ -b & a\end{Vmatrix}. \tag{21.6}$$

Comparing (21.4) with (21.6) we obtain (21.5). Furthermore, from (21.6) we get $2\,a = \operatorname{tr}\mathbf{A}$ and $a^2 + b^2 = \det\mathbf{A}$. Therefore the equation $L_{\boldsymbol{\eta}}(\mathbf{A}) = 0$ yields $L_{\boldsymbol{\eta}}(a) = 0$ and $L_{\boldsymbol{\eta}}(b) = 0$. Due to (18.2) then we have $a = \text{const}$ and $b = \text{const}$. $\square$

22. Constant curvature spaces.

If the system of equation (1.1) belongs to the case of general position and if its point symmetry algebra has maximal dimension (18.2), then curvature tensor for appropriate affine connection Γ has special structure (19.14), which corresponds to the spaces of constant sectional curvature $K = 1/(n-1)$ (see [24]). In special local coordinates metric tensor and components of connection for such spaces can be brought the special form. In order to construct such coordinates let's consider the following system of Pfaff equations for the components of covector field $\mathbf{u}$:

$$\nabla_i u_j = -u_i\,u_j - \frac{g_{ij}}{2(n-1)} + \frac{|\mathbf{u}|^2}{2}\,g_{ij},\quad i, j = 1, \ldots, n, \tag{22.1}$$

$$\text{where}\quad |\mathbf{u}|^2 = \sum_{r=1}^{n}\sum_{s=1}^{n} g^{rs}\,u_r\,u_s.$$

Complete compatibility of Pfaff equations (22.1) follows from (19.14) and (18.3). For the equations (22.1) let's consider Cauchy problem with zero initial data

$$u_j\Big|_{p=p_0} = 0 \tag{22.2}$$

at some fixed point p_0. Right hand side of the equations (22.1) do not vanish for $\mathbf{u} = 0$. Therefore the solution of the Cauchy problem (22.2) is unique covector field, which vanishes at the point p_0, but which is not identically zero.

By raising index with the use of metric $\mathbf{g}$ we convert covector field $\mathbf{u}$ into the vector field $\mathbf{u}$ with components

$$u^k = \sum_{j=1}^{n} g^{kj}\, u_j.$$

It satisfies the system of Pfaff equations derived from (22.1):

$$\nabla_i u^j = -u_i\, u^j - \frac{\delta_i^j}{2(n-1)} + \frac{|\mathbf{u}|^2}{2}\,\delta_i^j, \quad i,j = 1,\,\ldots\,,n. \tag{22.3}$$

By means of components of the field $\mathbf{u}$ we construct a tensor field $\mathbf{T}$ of type $(1,2)$. Its components are defined as follows:

$$T^k_{rs} = -u_r\,\delta^k_s - u_s\,\delta^k_r + u^k\, g_{rs}. \tag{22.4}$$

We use tensor $\mathbf{T}$ as a deformation tensor for connection Γ:

$$\bar\Gamma^k_{rs} = \Gamma^k_{rs} + T^k_{rs} = \Gamma^k_{rs} - u_r\,\delta^k_s - u_s\,\delta^k_r + u^k\, g_{rs}. \tag{22.5}$$

Its not difficult to calculate curvature tensor for new connection $\bar\Gamma$:

$$\bar R^k_{sij} = R^k_{sij} + \nabla_i T^k_{js} - \nabla_j T^k_{is} + \sum_{q=1}^{n} T^q_{js}\,T^k_{iq} - \sum_{q=1}^{n} T^q_{js}\,T^k_{iq}. \tag{22.6}$$

Substituting (19.14) and (22.4) into (22.6) and taking into account the equations (22.1) and (22.3) we get $\bar R^k_{sij} = 0$. Thus (22.5) is a plane (euclidean) connection.

Right hand side of the equations (22.1) is symmetric respective to i and j. Therefore $\nabla_i u_j = \nabla_j u_i$. Due to the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ this reduces to

$$\frac{\partial u_j}{\partial y^i} = \frac{\partial u_i}{\partial y^j}.$$

These relationships mean that covector field $\mathbf{u}$ is a gradient of some scalar field. One can find an explicit formula for this field. Let's take

$$f = \frac{1}{2(n-1)} + \frac{|\mathbf{u}|^2}{2}. \tag{22.7}$$

Function f in (22.7) is positive in some neighborhood of the point p_0 which defines initial data for Cauchy problem (22.2). From (22.1) we derive

$$\nabla_i f = -u_i\, f. \tag{22.8}$$

So covector field $\mathbf{u}$ is a gradient of the scalar field $\psi = -\ln f$.

Let's consider the metric $\bar{\mathbf{g}} = f^2\,\mathbf{g}$ conformally equivalent to the initial metric $\mathbf{g} = \mathbf{R}$. By means of direct calculations with the use of formula (18.4) we can see that metric connection for new metric $\bar{\mathbf{g}} = f^2\,\mathbf{g}$ coincides with (22.5).

Conclusion. *Metric $\bar{\mathbf{g}} = f^2\,\mathbf{g}$ is a flat (pseudoeuclidean) metric and there exist some local coordinates $y^1, \ldots, y^n$ on M such that components of the metric $\bar{\mathbf{g}}$ are constants and components of the connection $\bar{\Gamma}$ are zero.*

With respect to initial metric $\mathbf{g}$ such coordinates are called *conformally-euclidean* coordinates. In conformally-euclidean coordinates we have

$$\Gamma^k_{rs} = u_r\,\delta^k_s + u_s\,\delta^k_r - u^k\,g_{rs}. \tag{22.9}$$

These coordinates can be chosen so that constant matrix $\bar{g}_{ij}$ is brought to the canonical form. Then for initial metric $\mathbf{g}$ we get

$$\mathbf{g} = \sum_{i=1}^{n} \frac{\varepsilon_i\,dy^i \otimes dy^i}{f^2}, \tag{22.10}$$

where $\varepsilon_1 = \pm 1, \ldots, \varepsilon_n = \pm 1$. The number of pluses and minuses in the sequence $\varepsilon_1, \ldots, \varepsilon_n$ is defined by the signature of Ricci tensor $\mathbf{R} = \mathbf{g}$.

In order to find explicit form for the metric (22.10) we are only to find the explicit form of the function f in conformally-euclidean coordinates. Let's write (22.3) and (22.8) in these coordinates and take into account (22.7) and (22.9):

$$\frac{\partial f}{\partial y^i} = -u_i\,f, \qquad\qquad \frac{\partial u^j}{\partial y^i} = -u_i\,u^j - f\,\delta^j_i. \tag{22.11}$$

Now let's resolve the first equation (22.11) with respect to u_i and substitute the obtained expression for u_i into the second equation (2.11). This yields

$$\frac{\partial}{\partial y^i}\left(-\frac{u^j}{f}\right) = \delta^j_i.$$

This is the system of Pfaff equations which can be integrated in explicit form. Since initial data in (22.2) are zero, we get $u^j = -f\,y^j$. Here we assumed that fixed point p_0 is the origin for conformally-euclidean coordinates $y^1, \ldots, y^n$. Let's substitute $u^j = -f\,y^j$ into (22.7) and take into account formula (22.10) for metric. As a result for f we obtain explicit formula in conformally-euclidean coordinates:

$$f = \frac{1}{2(n-1)} + \frac{1}{2}\sum_{i=1}^{n} \varepsilon_i\,(y^i)^2. \tag{22.12}$$

So we got the proof of the following theorem first proved by Riemann.

Theorem 22.1. *In some neighborhood of any point p_0 on n-dimensional pseudoriemannian manifold of constant sectional curvature $K = 1/(n-1)$ there exist* *conformally-euclidean coordinates* $y^1, \ldots, y^n$ *such that metric tensor has the form* (22.10) *with the parameter* f *defined by* (22.12).

Theorem 22.2. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *If* $n \geqslant 3$, *there is a point transformation* (1.4) *bringing these equations to the form*

$$\frac{\partial y^i}{\partial t} = a\,\frac{\partial^2 y^i}{\partial x^2} + \sum_{r=1}^{n} a\,\frac{\varepsilon_r}{f}\,\frac{\partial y^r}{\partial x}\left(y^i\,\frac{\partial y^r}{\partial x} - 2\,y^r\,\frac{\partial y^i}{\partial x}\right), \tag{22.13}$$

where $i = 1, \ldots, n$, $a = \mathrm{const}$, $\varepsilon_1 = \pm 1, \ldots, \varepsilon_n = \pm 1$, *and* $f = f(y^1, \ldots, y^n)$ *is a function defined by* (22.12).

The equations (22.13) arise if we substitute (22.9) into (1.1) and take into account $u^j = -f\,y^j$. In two-dimensional case $n = 2$ appropriate equations appear to be more huge due to more huge formula for $\mathbf{A}$:

$$\begin{aligned}\frac{\partial y^1}{\partial t} &= a\,\frac{\partial^2 y^1}{\partial x^2} + \sum_{r=1}^{n} a\,\frac{\varepsilon_r}{f}\,\frac{\partial y^r}{\partial x}\left(y^1\,\frac{\partial y^r}{\partial x} - 2\,y^r\,\frac{\partial y^1}{\partial x}\right) + \\ &+ b\,\varepsilon_1\,\frac{\partial^2 y^2}{\partial x^2} + \sum_{r=1}^{n} b\,\varepsilon_1\,\frac{\varepsilon_r}{f}\,\frac{\partial y^r}{\partial x}\left(y^2\,\frac{\partial y^r}{\partial x} - 2\,y^r\,\frac{\partial y^2}{\partial x}\right),\end{aligned} \tag{22.14}$$

$$\begin{aligned}\frac{\partial y^2}{\partial t} &= a\,\frac{\partial^2 y^2}{\partial x^2} + \sum_{r=1}^{n} a\,\frac{\varepsilon_r}{f}\,\frac{\partial y^r}{\partial x}\left(y^2\,\frac{\partial y^r}{\partial x} - 2\,y^r\,\frac{\partial y^2}{\partial x}\right) - \\ &- b\,\varepsilon_2\,\frac{\partial^2 y^1}{\partial x^2} - \sum_{r=1}^{n} b\,\varepsilon_2\,\frac{\varepsilon_r}{f}\,\frac{\partial y^r}{\partial x}\left(y^1\,\frac{\partial y^r}{\partial x} - 2\,y^r\,\frac{\partial y^1}{\partial x}\right).\end{aligned} \tag{22.15}$$

Theorem 22.3. *Let* (1.1) *be a system of equations belonging to the case of general position such that its algebra of point symmetries has maximal dimension* (18.2). *If* $n = 2$, *then there is a point transformation* (1.4) *bringing these equations to the form* (22.14) *and* (22.15), *where* $a = \mathrm{const}$, $b = \mathrm{const}$, $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, *and* $f = f(y^1, y^2)$ *is a function defined by* (22.12).

23. Case of one equation.

For $n = 1$ we have only one equation in the system (1.1). Operator field $\mathbf{A}$ and affine connection Γ have only one component:

$$\frac{\partial y}{\partial t} = A\left(\frac{\partial^2 y}{\partial x^2} + \Gamma\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial x}\right). \tag{23.1}$$

Any affine connection on one-dimensional manifold has identically zero curvature tensor. Therefore it is flat and its Ricci tensor is zero. This means $m = 0$. Any equation (23.1) belongs to the case of maximal degeneration. For the dimension of its symmetry algebra we have an estimate $\dim(\mathcal{G}) \leqslant 2$, which follows from theorem 6.1.

Theorem 23.1. *For the equation* (23.1) *with the symmetry algebra of maximal dimension* $\dim(\mathcal{G}) = 2$ *there exist a point transformation bringing it to the form*

$$\frac{\partial y}{\partial t} = a\,\frac{\partial^2 y}{\partial\,x^2}, \quad \textit{where} \quad a = \mathrm{const}\,.$$

Note. According to theorem 17.2 if system of equations (1.1) belongs to m-th case of intermediate degeneration and possess the algebra of point symmetries of maximal dimension (9.1), then it admit the variable separation resulting in smaller subsystem of m equations. Such subsystem belongs to the case of general position and has the algebra of point symmetries of maximal dimension (18.2), where $n = m$. To this system we can apply one of the theorems 22.2, 22.3, or 23.1.

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