INVARIANT INTEGRABILITY CRITERION FOR THE EQUATIONS OF HYDRODYNAMICAL TYPE.

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Abstract. Invariant integrability criterion for the equations of hydrodynamical type is found. This criterion is written in the form of vanishing for some tensor which is derived from the velocities matrix of hydrodynamical equations.

1. Introduction.

Systems of quasilinear partial differential equations of the first order arise in different models describing the motion of continuous media. Special subclass of such systems is known as a systems of equations of *hydrodynamical type*. In spatially one-dimensional case they are written as follows

$$u^i_t=\sum_{j=1}^{n}A^i_j(\mathbf{u})u^j_x,\ \text{ where } i=1,\dots,n \tag{1.1}$$

Among the systems (1.1) man can consider special subclass of systems possessing the *Riemann invariants*. These are the systems which can be transformed to the diagonal form

$$u^i_t=\lambda_i(\mathbf{u})u^i_x,\ \text{ where } i=1,\dots,n \tag{1.2}$$

by means of so called *point transformations*

$$\tilde u^i=\tilde u^i(u^1,\dots,u^n),\ \text{ where } i=1,\dots,n \tag{1.3}$$

System of equations (1.1) is called the *hydrodynamically integrable system* if it has the continuous set of hydrodynamical symmetries (or hydrodynamical conservation laws) parameterized by n arbitrary functions of one variable. For diagonal systems (1.2) with mutually distinct characteristic velocities ($\lambda_i\neq\lambda_j$) one has the well-developed theory of integration (see reviews [1] and [2]). As it was shown in [2] the diagonal system (1.2) is integrable if and only if the following condition is satisfied

$$\partial_i\left(\frac{\partial_j\lambda_k}{\lambda_j-\lambda_k}\right)=\partial_j\left(\frac{\partial_i\lambda_k}{\lambda_i-\lambda_k}\right),\ \text{ where } i\neq k\neq j \tag{1.4}$$

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where $\partial_i = \partial/\partial u^i$ and $\partial_j = \partial/\partial u^j$. Such systems are called *semi-Hamiltonian systems*, for the property (1.4) itself in Russian papers the term *semihamiltonity* is used. When the diagonal system (1.2) possess this property it can be integrated by means of "generalized hodograph method" (details see in [2])

In section 2 of this paper we consider the problem of hydrodynamical integrability for the systems of equations (1.1) with the velocities matrix $A^i_j(\mathbf{u})$ of general position i.e. eigenvalues of which are mutually distinct. There we managed to prove the following fact: each hydrodynamically integrable system (1.1) with the matrix of general position is necessarily diagonalizable[1]. Summing up this fact with the results of [2] we may state the following theorem

Theorem 1. *System of equations* (1.1) *with the matrix of general position is hydrodynamically integrable if and only if it is diagonalizable and semi-Hamiltonian.*

Theorem 1 shows that the study of diagonal equations (1.2) is very important. But it doesn't exclude the necessity of study of general equations (1.1). Indeed the integrability test for the equations (1.1) of general position according to the theorem 1 should include 3 steps

(1) test of diagonalizability,
(2) diagonalization by means of the point transformation (1.3),
(3) test of semihamiltonity (1.4).

First of these steps is implemented by means of invariant geometrical criterion from [3]. This criterion consists in vanishing of Haantjes's tensor derived from the velocities matrix of the equations in question. This test was first applied to the equations (1.1) in [4].

Next step is an actual diagonalization. To pass this step one should calculate the eigenvalues of the matrix $A^i_j(\mathbf{u})$, find its eigenvectors, properly normalize them and then one should solve some system of ordinary differential equations defining the transformation (1.2). Because of this step the total integrability test may be absolutely inefficient since the case when the system of differential equations is explicitly solvable is very rare event. However if we find such solution the third step may have only the difficulties in calculations.

The presence of nonefficient step in the above integrability test of the equations (1.1) is due to the absence of the of invariant criterion for testing the semihamiltonity for these equations. The main goal of this paper is to eliminate this essential fault of the theory of such equations. As it was noted in [1]: It was Riemann who first recognized that the theory of the equations (1.1) is the theory of tensors since the components of matrix $A^i_j(\mathbf{u})$ are transformed as the components of tensor under the point transformations (1.3). Therefore it is natural to expect that the semihamiltonity relationships (1.4) can be rewritten in an invariant tensorial form. In the section 4 of this paper we construct the tensor, vanishing of which is equivalent to (1.4). So invariant integrability criterion for the equations of hydrodynamical type is obtained. New integrability test now is absolutely efficient. It includes two steps

(1) test of vanishing the tensor of Haantjes,
(2) test of vanishing the semihamiltonity tensor.

In order to construct the semihamiltonity tensor we use the theory developed by Froelicher and Nijenhuis in [5] and [6]. This theory in brief is given in section

[1]May be this is not new fact but we couldn't find it anywhere.

3. According to the theory of Froelicher and Nijenhuis each smooth manifold is equipped with some Lie superalgebra of tensor fields of type $(1, p)$. Note that the paper [6] was written in 1956 but unfortunately it wasn't known to specialists since for example in [7] the paper of Martin of 1959 is quoted as the first paper in supermathematics.

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2. Hydrodynamical integrability.

Let's take the system of equations of hydrodynamical type (1.1) and let's add to it another such system with the dynamics by the variable τ

$$u^i_\tau = \sum_{j=1}^{n} B^i_j(\mathbf{u})u^j_x, \text{ where } i = 1, \ldots, n \tag{2.1}$$

Definition 1. System of equations (2.1) is called the hydrodynamical symmetry for the equations (1.1) if the equations (1.1) and (2.1) are compatible.

Now we shall study the question about the existence and the number of hydrodynamical symmetries for the system of equations (1.1). Let the equations (1.1) and (2.1) be compatible. Their compatibility conditions are written in form of the relationships

$$\sum_{s=1}^{n} A^i_s B^s_j = \sum_{s=1}^{n} B^i_s A^s_j \tag{2.2}$$

$$\begin{gathered} \sum_{s=1}^{n}\bigl(\partial_s A^k_i B^s_j + \partial_s A^k_j B^s_i + \partial_i B^s_j A^k_s + \partial_j B^s_i A^k_s\bigr) = \\ = \sum_{s=1}^{n}\bigl(\partial_s B^k_i A^s_j + \partial_s B^k_j A^s_i + \partial_i A^s_j B^k_s + \partial_j A^s_i B^k_s\bigr) \end{gathered} \tag{2.3}$$

The relationship (2.2) means that the matrices the systems (1.1) and (2.1) are commuting

$$\mathbf{AB} = \mathbf{BA} \tag{2.4}$$

Second relationship (2.3) also can be written in an invariant form. In order to do it let's contract (2.3) with $X^i X^j$, where $X^1, \ldots, X^n$ are the components of some arbitrary vector field $\mathbf{X}$.

$$[\mathbf{AX}, \mathbf{BX}] - \mathbf{A}[\mathbf{X}, \mathbf{BX}] - \mathbf{B}[\mathbf{AX}, \mathbf{X}] = 0 \tag{2.5}$$

This result can be stated as a theorem.

Theorem 2. *System of the equations* (2.1) *is a hydrodynamical symmetry for the system* (1.1) *if and only if for any vector field* $\mathbf{X}$ *the relationships* (2.4) *and* (2.5) *hold.*

Let the operator field $\mathbf{A} = \mathbf{A}(\mathbf{u})$ from (1.1) have n mutually distinct eigenvalues $\lambda_i = \lambda_i(\mathbf{u})$. Through $\mathbf{X}_1, \ldots, \mathbf{X}_n$ we denote the frame formed by eigenvectors of operator $\mathbf{A}$. The choice of eigenvectors is not unique, there is the gauge arbitrariness in n scalar factors

$$\mathbf{X}_i(\mathbf{u}) \longrightarrow f_i(\mathbf{u})\mathbf{X}_i(\mathbf{u}), \;\text{ where } f_i \neq 0 \tag{2.6}$$

For the sake of brevity we introduce the following notations for the Lie derivatives along the vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_n$

$$L_i = L_{\mathbf{X}_i} \tag{2.7}$$

Mutual commutators of the vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are convenient to be expanded in the frame formed by these fields

$$L_i\mathbf{X}_j = [\mathbf{X}_i, \mathbf{X}_j] = \sum_{k=1}^{n} c^k_{ij}\mathbf{X}_k \tag{2.8}$$

Parameters $c^k_{ij} = c^k_{ij}(\mathbf{u})$ in (2.8) are to be called the *structural scalars* of the frame $\mathbf{X}_1, \ldots, \mathbf{X}_n$. The term *structural constants* doesn't suit since c^k_{ij} depend on the point $\mathbf{u}$.

From the algebra we know that the matrix $\mathbf{B}$ is commuting with the matrix $\mathbf{A}$ having mutually distinct eigenvalues then these two matrices are simultaneously diagonalized in the frame $\mathbf{X}_1, \ldots, \mathbf{X}_n$. Therefore any operator $\mathbf{B}$ satisfying (2.4) is completely defined by its eigenvalues $\mu_i = \mu_i(\mathbf{u})$. Note that from (2.5) we have the relationship

$$\begin{aligned}&[\mathbf{AX}, \mathbf{BY}] + [\mathbf{AY}, \mathbf{BX}] - \mathbf{A}[\mathbf{X}, \mathbf{BY}] - \\ &- \mathbf{A}[\mathbf{Y}, \mathbf{BX}] - \mathbf{B}[\mathbf{AX}, \mathbf{Y}] - \mathbf{B}[\mathbf{AY}, \mathbf{X}] = 0\end{aligned}$$

which holds for two arbitrary vector fields $\mathbf{X}$ and $\mathbf{Y}$. Let's substitute $\mathbf{X} = \mathbf{X}_i$ and $\mathbf{Y} = \mathbf{X}_j$ into the above relationship. As a result we obtain that it is equivalent to the pair of sets of relationships. First set is algebraic with respect to the eigenvalues μ_i of the matrix $\mathbf{B}$

$$\begin{aligned}&c^i_{jk}(\lambda_j - \lambda_k)\mu_i + c^i_{jk}(\lambda_k - \lambda_i)\mu_j + \\ &\quad + c^i_{jk}(\lambda_i - \lambda_j)\mu_k = 0, \;\text{ for } i \neq j, j \neq k, k \neq i\end{aligned} \tag{2.9}$$

Second set contains the partial differential equations with respect to μ_i

$$L_i\mu_j = \lambda_{ij}\frac{\mu_i - \mu_j}{\lambda_i - \lambda_j}, \;\text{ for } i \neq j \tag{2.10}$$

Here and everywhere below we use the notations $\lambda_{ij} = L_i\lambda_j$ in terms of Lie derivatives (2.7).

System of differential equations (2.10) is overdetermined. When it is compatible the maximal degree of arbitrariness for its solutions is n functions of one variables. Let this degree of arbitrariness be actually realized. Then

$$\mu_i = \mu_i(f_1, \dots, f_n, \mathbf{u}) \tag{2.11}$$

Let's substitute (2.11) into (2.9). This leads to the functional dependence for the parameters $f_1(z_1), \dots, f_n(z_n)$ from (2.11), which contradicts their arbitrariness. Therefore the relationships (2.9) should be trivial. For the structural scalars (2.8) this gives

$$c^k_{ij} = 0, \ \text{ for } i \neq j, j \neq k, k \neq i \tag{2.12}$$

The relationships (2.12) have the important consequences which are due to the following lemma.

Lemma 1. *The linear operator of the general position* $\mathbf{A} = \mathbf{A}(\mathbf{u})$ *is diagonalizable by means of the transformation* (1.3) *if and only if the relationships* (2.12) *hold for the frame of its eigenvectors.*

We give the sketch of proof of this lemma. Operator $\mathbf{A}(\mathbf{u})$ is diagonal in the frame of its eigenvectors. For $\mathbf{A}(\mathbf{u})$ to be diagonalizable by the transformation (1.3) this frame should be the coordinate frame i.e. structural scalars of this frame should be identically zero $c^k_{ij} = 0$. The relationship (2.12) provides vanishing most of these scalars. One can reach vanishing the rest of these scalars by use of the gauge arbitrariness (2.6).

Because of lemma 1 the further analysis of the compatibility conditions for the equations (2.10) becomes unnecessary. For the diagonal systems (1.2) such analysis was done by S.P.Tsarev in [2]. Note only that as result of such analysis we add to (2.12) the following relationships

$$\begin{aligned} &L_i\left(\frac{\lambda_{jk}}{\lambda_j - \lambda_k}\right) - L_j\left(\frac{\lambda_{ik}}{\lambda_i - \lambda_k}\right) + \\ &+ \frac{c^j_{ji}\lambda_{jk}}{\lambda_j - \lambda_k} - \frac{c^i_{ij}\lambda_{ik}}{\lambda_i - \lambda_k} = 0 \end{aligned} \tag{2.13}$$

which hold for $i \neq j, j \neq k, k \neq i$. The relationships (2.13) are the same as the semihamiltonity relationships (1.4) but written in the frame of eigenvectors of diagonalizable operator $\mathbf{A}(\mathbf{u})$. The above considerations prove the theorem 1 in the following form.

Theorem 3. *System of the equations* (1.1) *with the matrix of general position possess the continuous set of hydrodynamical symmetries with functional arbitrariness given by* n *functions of one variable if and only if it is diagonalizable and semi-Hamiltonian.*

Now let's study the similar question about the conservation laws for (1.1). On the set of their solutions we define the integral functionals of the following form

$$F = \int f(\mathbf{u})dx \tag{2.14}$$

Functional (2.14) is called the hydrodynamical conservation law or the first integral for the equations (1.1) if $\dot{F} = 0$ when time derivative $\dot{F}$ is calculated according to the dynamics given by (1.1). This derivative is the following integral functional

$$G = \dot{F} = \int \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i f A^i_j u^j_x \right) dx \tag{2.15}$$

From vanishing the functional (2.15) we get the vanishing of its variational derivatives

$$\frac{\delta G}{\delta u^i} = \sum_{j=1}^{n} \sum_{k=1}^{n} \bigl(\partial_i(\partial_k f A^k_j) - \partial_j(\partial_k A^k_i)\bigr) u^j_x = 0$$

This leads to the following relationship for the density $f(\mathbf{u})$ of the primary functional (2.14)

$$\sum_{k=1}^{n} \bigl(\partial_i(\partial_k f A^k_j) - \partial_j(\partial_k A^k_i)\bigr) = 0 \tag{2.16}$$

It is equivalent to the existence of the function $T(\mathbf{u})$ such that

$$G = \int \sum_{i=1}^{n} \sum_{j=1}^{n} (\partial_i f A^i_j u^j_x) dx = \int \frac{\partial T}{\partial x} dx \tag{2.17}$$

Because of (2.17) the condition (2.16) is exactly the condition of vanishing the functional $G = \dot{F} = 0$. In order to write the relationships (2.16) in an invariant form let's choose two arbitrary vector fields $\mathbf{X}$ and $\mathbf{Y}$. After contracting (2.16) with X^i and Y^j we can write the result of such contraction through the Lie derivatives

$$\bigl(L_{\mathbf{X}} L_{\mathbf{AY}} - L_{\mathbf{Y}} L_{\mathbf{AX}} - L_{\mathbf{A}[\mathbf{X},\mathbf{Y}]}\bigr) f = 0 \tag{2.18}$$

This result can be stated in form of the following theorem.

Theorem 4. *Integral functional* (2.14) *is hydrodynamical conservation law for the system of equations* (1.1) *if and only if for any choice of vector fields* $\mathbf{X}$ *and* $\mathbf{Y}$ *the equations* (2.18) *hold.*

The relationship (2.18) is the system of differential equations with respect to the unknown function $f(\mathbf{u})$. One should investigate it for the compatibility. Let's denote $L_i f = \varphi_i$. Substituting frame vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ for $\mathbf{X}$ and $\mathbf{Y}$ into the relationships (2.18) we get the following equations

$$L_i \varphi_j = \sum_{k=1}^{n} B^k_{ij} \varphi_k, \ \text{ for } i \neq j \tag{2.19}$$

where the functions B^k_{ij} are defined by the formulae

$$B^k_{ij} = c^k_{ij} \frac{\lambda_k - \lambda_i}{\lambda_j - \lambda_i} + \frac{\lambda_{ji} \delta^k_i}{\lambda_j - \lambda_i} - \frac{\lambda_{ij} \delta^k_j}{\lambda_j - \lambda_i} \tag{2.20}$$

The equations (2.19) are analogous to the equations (2.10). When they are compatible their solutions have the arbitrariness in n functions of one variable. Let such arbitrariness be actually realized. We look for the differential consequences of the equations (2.19). Among them we find the following relationships

$$B^i_{jk} L_i\varphi_i - B^j_{ik} L_j\varphi_j - c^k_{ij} L_k\varphi_k = -\sum_{q=1}^{n} R^q_{kij}\varphi_q \tag{2.21}$$

which hold for $i \neq j, j \neq k, k \neq i$. The values of R^q_{kij} for $i \neq j, j \neq k, k \neq i$ are calculated according to the formula

$$\begin{aligned} R^q_{kij} = L_i B^q_{jk} + \sum_{s\neq i}^{n} B^q_{is} B^s_{jk} - \\ - L_j B^q_{ik} - \sum_{s\neq j}^{n} B^q_{js} B^s_{ik} - \sum_{s\neq k}^{n} c^s_{ij} B^q_{sk} \end{aligned} \tag{2.22}$$

The derivatives Ł$_i\varphi_i$, Ł$_j\varphi_j$ and Ł$_k\varphi_k$ aren't defined by the equations (2.19). This gives the arbitrariness in n functions for the solutions of (2.19). When they are nontrivial the relationships (2.21) define the functional dependence between these derivatives. Therefore they diminish the degree of arbitrariness. In case of maximal arbitrariness the relationships (2.12) should be trivial

$$B^i_{jk} = B^j_{ik} = c^k_{ij} = 0, \ \text{ for } i \neq j, j \neq k, k \neq i \tag{2.23}$$

From (2.23) due to the lemma 1 we get the diagonalizability of the operator $\mathbf{A}(\mathbf{u})$ by means of point transformations from (1.3). Due to (2.20) the equations (2.19) are rewritten as follows

$$L_i\varphi_j = -\frac{\lambda_{ji}\varphi_i - \lambda_{ij}\varphi_j}{\lambda_i - \lambda_j} + c^j_{ij}\varphi_j, \ \text{ for } i \neq j \tag{2.24}$$

The compatibility conditions for (2.24) are defined by the quantities from (2.22) as $R^q_{kij} = 0$. On taking into account (2.23) these compatibility conditions are exactly coincide with (2.13). In spite of the fact that the equations (2.10) and (2.24) are different their compatibility conditions are the same and have the form of semihamiltonity condition written in the frame of eigenvectors of the operator $\mathbf{A}(\mathbf{u})$. The above considerations prove the following version of the theorem 1.

Theorem 5. *System of the equations* (1.1) *with the matrix of general position possess the continuous set of conservation laws parameterized by* n *functions of one variable if and only if it is diagonalizable and semi-Hamiltonian.*

3. The Froelicher-Nijenhuis bracket and the Lie superalgebra of vector-valued differential forms.

Let $\mathbf{A}$ be the tensor field of the type $(1, p)$ and let it be skew symmetric in covariant components. Then $\mathbf{A}$ defines the vector-valued p-form $\mathbf{A} = \mathbf{A}(\mathbf{X}_1, \ldots, \mathbf{X}_p)$. Here $\mathbf{X}_1, \ldots, \mathbf{X}_p$ are some arbitrary vector fields. Let $\mathbf{B}$ be the second tensor field of the type $(1,q)$ which define vector valued q-form $\mathbf{B}(\mathbf{X}_1,\dots,\mathbf{X}_q)$. We shall call $\mathbf{A}$ and $\mathbf{B}$ the vector fields of rank 1 if they are of the following form

$$\mathbf{A}=\mathbf{a}\otimes\alpha \qquad \mathbf{B}=\mathbf{b}\otimes\beta \tag{3.1}$$

where $\mathbf{a}$ and $\mathbf{b}$ are vector fields while α and β are the differential forms. For the tensor fields of the form (3.1) we define the pairing $\{\mathbf{A},\mathbf{B}\}$ (it is known as Froelicher-Nijenhuis bracket)

$$\begin{aligned}\{\mathbf{A},\mathbf{B}\}&=[\mathbf{a},\mathbf{b}]\otimes\alpha\wedge\beta-\mathbf{a}\otimes L_{\mathbf{b}}\alpha\wedge\beta+\mathbf{b}\otimes\alpha\wedge L_{\mathbf{a}}\beta+\\&+(-1)^p\mathbf{a}\otimes\iota_{\mathbf{b}}\alpha\wedge d\beta+(-1)^p\mathbf{b}\otimes d\alpha\wedge\iota_{\mathbf{a}}\beta\end{aligned} \tag{3.2}$$

Via $\iota_{\mathbf{a}}$ and $\iota_{\mathbf{b}}$ in formula (3.2) we denote the differentiations of substitution. For the r-form ω and for the vector field $\mathbf{c}$ the expression $\iota_{\mathbf{c}}\omega$ is a $r-1$-form

$$\iota_{\mathbf{c}}\omega(\mathbf{X}_1,\dots,\mathbf{X}_{r-1})=r\omega(\mathbf{c},\mathbf{X}_1,\dots,\mathbf{X}_{r-1})$$

The operation $\iota_{\mathbf{c}}$ is also known as inner product with respect to the vector field $\mathbf{c}$ (see [8]).

Theorem 6. *The bracket $\{\mathbf{A},\mathbf{B}\}$ defined for the tensor fields $\mathbf{A}$ and $\mathbf{B}$ of rank 1 by the formula* (3.2) *is uniquely continued for the arbitrary tensor fields of the types* $(1,p)$ *and* $(1,q)$ *skew symmetric in their covariant components.*

PROOF. Each tensor field of the type $(1,p)$ can be written as a sum of tensor fields of rank one as follows

$$\mathbf{A}=\sum_i\mathbf{A}_i=\sum_i\mathbf{a}_i\otimes\alpha_i \tag{3.3}$$

The analogous formula can be written for the field $\mathbf{B}$. Therefore the bracket (3.2) for the arbitrary $\mathbf{A}$ and $\mathbf{B}$ can be redefined as

$$\{\mathbf{A},\mathbf{B}\}=\sum_i\sum_j\{\mathbf{A}_i,\mathbf{B}_j\} \tag{3.4}$$

However the expansion (3.3) is not unique. Therefore the definition $\{\mathbf{A},\mathbf{B}\}$ by means of (3.4) should be tested for the correctness. The arbitrariness in the expansion (3.3) for $\mathbf{A}$ is defined by the following identities in tensor algebra

$$\begin{aligned}&(\mathbf{a}+\tilde{\mathbf{a}})\otimes\alpha=\mathbf{a}\otimes\alpha+\tilde{\mathbf{a}}\otimes\alpha\\&\mathbf{a}\otimes(\alpha+\tilde{\alpha})=\mathbf{a}\otimes\alpha+\mathbf{a}\otimes\tilde{\alpha}\end{aligned} \tag{3.5}$$

$$(f\mathbf{a})\otimes\alpha=\mathbf{a}\otimes(f\alpha) \tag{3.6}$$

where f is an arbitrary scalar field. The arbitrariness due to (3.5) does not influence to the value of bracket (3.4) since the relationship (3.2) is additive with respect to $\mathbf{a}$ and α. Let's ensure that the arbitrariness due to (3.6) also doesn't make the influence to the value of $\{\mathbf{A},\mathbf{B}\}$. In order to do it we calculate this bracket by (3.2) first for $\mathbf{A}=(f\mathbf{a})\otimes\alpha$ then for $\mathbf{A}=\mathbf{a}\otimes(f\alpha)$ and after all we compare the results. All these calculations are based on the following formulae from [8]

$$\begin{aligned}&[f\mathbf{a},\mathbf{b}]=f[\mathbf{a},\mathbf{b}]-\mathbf{a}L_{\mathbf{b}}f &\qquad& L_{f\mathbf{a}}\beta=fL_{\mathbf{a}}\beta+df\wedge\iota_{\mathbf{a}}\beta\\&L_{\mathbf{b}}(f\alpha)=fL_{\mathbf{b}}\alpha+L_{\mathbf{b}}f\alpha &\qquad& \iota_{f\mathbf{a}}\beta=f\iota_{\mathbf{a}}\beta\end{aligned}$$

Since these calculations are standard we did not write them here. Theorem is proved □

Note that for $p=q=0$ the bracket (3.2) coincides with the ordinary commutator of vector fields. For the arbitrary values of p and q the algebraic properties of this bracket are given by the following theorem.

Theorem 7. *The bracket* $\{\mathbf{A},\mathbf{B}\}$ *for the tensor fields of rank 1 defined by* (3.2) *and then generalized by* (3.4) *satisfies the relationships*

$$\{\mathbf{A},\mathbf{B}\}+(-1)^{pq}\{\mathbf{B},\mathbf{A}\}=0$$
$$\{\{\mathbf{A},\mathbf{B}\},\mathbf{C}\}(-1)^{rp}+\{\{\mathbf{B},\mathbf{C}\},\mathbf{A}\}(-1)^{pq}+\{\{\mathbf{C},\mathbf{A}\},\mathbf{B}\}(-1)^{qr}=0$$

because of which it defines the structure of graded Lie superalgebra in tensor fields of type $(1,m)$ *skew symmetric in covariant components.*

Let $\mathbf{A}$ and $\mathbf{B}$ be tensor fields of the type (1.1) i.e. operator fields. Tensor field $\mathbf{S}=2\{\mathbf{A},\mathbf{B}\}$ is the vector valued 2-form. Its values may be calculated by the following formula

$$\begin{aligned}\mathbf{S}(\mathbf{X},\mathbf{Y})&=[\mathbf{AX},\mathbf{BY}]+[\mathbf{BX},\mathbf{AY}]+\\&+\mathbf{AB}[\mathbf{X},\mathbf{Y}]+\mathbf{BA}[\mathbf{X},\mathbf{Y}]-\mathbf{A}[\mathbf{X},\mathbf{BY}]-\\&-\mathbf{A}[\mathbf{BX},\mathbf{Y}]-\mathbf{B}[\mathbf{X},\mathbf{AY}]-\mathbf{B}[\mathbf{AX},\mathbf{Y}]\end{aligned}\tag{3.7}$$

Tensor $\mathbf{S}$ is known as the torsion of Nijenhuis for the operator fields $\mathbf{A}$ and $\mathbf{B}$ (see [5], [6] and [8]).

4. The construction of semihamiltonity tensor.

First let's recall the classical invariant criterion of diagonalizability for the operator field $\mathbf{A}$. We mentioned this criterion in section 1 (see also [3], [4] and [6]). Let's consider the particular form of tensor (3.7)

$$\mathbf{N}=\{\mathbf{A},\mathbf{A}\}$$

It is usually called the tensor of Nijenhuis. From (3.7) we obtain

$$\begin{aligned}\mathbf{N}(\mathbf{X},\mathbf{Y})&=[\mathbf{AX},\mathbf{AY}]+\mathbf{A}^2[\mathbf{X},\mathbf{Y}]-\\&-\mathbf{A}[\mathbf{X},\mathbf{AY}]-\mathbf{A}[\mathbf{AX},\mathbf{Y}]\end{aligned}\tag{4.1}$$

Tensor of Haantjes is defined via the tensor of Nijenhuis (4.1) according to the formula

$$\begin{aligned}\mathbf{H}(\mathbf{X},\mathbf{Y})&=\mathbf{N}(\mathbf{AX},\mathbf{AY})+\mathbf{A}^2\mathbf{N}(\mathbf{X},\mathbf{Y})-\\&-\mathbf{AN}(\mathbf{X},\mathbf{AY})-\mathbf{AN}(\mathbf{AX},\mathbf{Y})\end{aligned}\tag{4.2}$$

It has the same type as the tensor $\mathbf{N}$. It is also the vector-valued skew symmetric 2-form.

Theorem 8 (criterion of diagonalizability)**.** *The operator* $\mathbf{A}(\mathbf{u})$ *of general position with mutually distinct eigenvalues is diagonalizable by means of point transformations* (1.3) *if and only if its tensor of Haantjes* (4.2) *is identically zero.*

The criterion of diagonalizability in form of this theorem was first proved in [3]. It was applied to the systems of equations (1.1) in [4].

PROOF. Because of skew symmetry of bilinear form (4.2) it's enough to test vanishing this form only for vector fields $\mathbf{X} = \mathbf{X}_i$ and $\mathbf{Y} = \mathbf{X}_j$ from the frame of eigenvectors of the operator $\mathbf{A}(\mathbf{u})$ with $i \neq j$. By direct calculations we obtain

$$\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j) = \sum_{k=1}^{n} (\lambda_i - \lambda_k)^2 (\lambda_j - \lambda_k)^2 c^k_{ij} \mathbf{X}_k \tag{4.3}$$

Because of (4.3) the equality $\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j) = 0$ is equivalent to (2.12). Then we are only to apply the lemma 1. Criterion is proved. □

Practical use of this criterion for the testing the diagonalizability is based on the following formulae for the components of tensors $\mathbf{N}$ and $\mathbf{H}$ expressing them through the components of matrix $\mathbf{A}(\mathbf{u})$

$$N^k_{ij} = \sum_{s=1}^{n} \left(A^s_i \partial_s A^k_j - A^s_j \partial_s A^k_i + A^k_s \partial_j A^s_i - A^k_s \partial_i A^s_j \right) \tag{4.4}$$

$$\begin{aligned} H^k_{ij} = \sum_{s=1}^{n} \sum_{r=1}^{n} \bigl(& A^k_s A^s_r N^r_{ij} - \\ & - A^k_s N^s_{rj} A^r_i - A^k_s N^s_{ir} A^r_j + N^k_{sr} A^s_i A^r_j \bigr) \end{aligned} \tag{4.5}$$

Let $\mathbf{B}$ be the operator field i.e the tensor field of type $(1, 1)$ and let $\mathbf{Q}$ be the skew symmetric tensor field of type $(1, 2)$. Through $\mathbf{K}$ we denote the Froelicher–Nijenhuis bracket of these two fields $\mathbf{K} = 3\{\mathbf{Q}, \mathbf{B}\}$. For the 3-form $\mathbf{K}$ we have

$$\begin{aligned} &\mathbf{K}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{B}[\mathbf{X}, \mathbf{Q}(\mathbf{Y}, \mathbf{Z})] - [\mathbf{B}\mathbf{X}, \mathbf{Q}(\mathbf{Y}, \mathbf{Z})] + \\ &+ \mathbf{B}\mathbf{Q}(\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]) + \mathbf{Q}(\mathbf{X}, \mathbf{B}[\mathbf{Y}, \mathbf{Z}]) - \mathbf{Q}(\mathbf{X}, [\mathbf{B}\mathbf{Y}, \mathbf{Z}]) - \\ &- \mathbf{Q}(\mathbf{X}, [\mathbf{Y}, \mathbf{B}\mathbf{Z}]) + \dots \end{aligned} \tag{4.6}$$

Dots in (4.6) denote 12 summand that can be obtained from explicitly written summands in (4.6) by means of the cyclic transposition of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$.

Starting with deriving the invariant semihamiltonity criterion we should note that everywhere above (see theorems 1, 3 and 5) the semihamiltonity comes together with diagonalizability. It doesn't play the separate role. Therefore we will use the following scheme of action: we will construct the tensor vanishing of which gives the equations (1.4) after bringing this tensor to the coordinates where the matrix $\mathbf{A}(\mathbf{u})$ is diagonal. The equations (1.4) are rational. Let's rewrite them in polynomial form. In order to do it we introduce the following quantities

$$\begin{aligned} \alpha^k_{kij} = &-(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)(\lambda_j - \lambda_k) \partial_{ij} \lambda_k - \\ &- (\lambda_i - \lambda_j)(\lambda_i + \lambda_j - 2\lambda_k) \partial_i \lambda_k \partial_j \lambda_k + \\ &+ (\lambda_i - \lambda_k)^2 \partial_i \lambda_j \partial_j \lambda_k - (\lambda_j - \lambda_k)^2 \partial_j \lambda_i \partial_i \lambda_k \end{aligned} \tag{4.7}$$

The semihamiltonity (1.4) then is written as the condition of vanishing the quantities (4.7)

$$\alpha^k_{kij} = 0, \ \text{ for } i \neq k \neq j \tag{4.8}$$

Partial derivatives in (4.7) as in (1.4) are calculated with respect to the variables $u^1, \ldots, u^n$ for which the matrix $\mathbf{A}(\mathbf{u})$ is diagonal. The frame of eigenvectors $\mathbf{X}^1, \ldots, \mathbf{X}^n$ is chosen to be the coordinate frame for such variables.

Using the Froelicher–Nijenhuis bracket we construct the tensor $\mathbf{K}$ from the matrix $\mathbf{A}$ as follows

$$\mathbf{K} = 3\{\{\mathbf{A}, \mathbf{A}\}, \mathbf{A}^2\} = 3\{\mathbf{N}, \mathbf{A}^2\} \tag{4.9}$$

Tensor $\mathbf{K}$ defines the vector-valued 3-form the values of which for the frame vectors can be calculated according to (4.6). As a result from (4.9) we have

$$\mathbf{K}(\mathbf{X}_k, \mathbf{X}_i, \mathbf{X}_j) = K^k_{kij}\mathbf{X}_k + K^i_{kij}\mathbf{X}_i + K^j_{kij}\mathbf{X}_j \tag{4.10}$$

(there is summation by i, j, k here). We are interested only in one group of components of tensor $\mathbf{K}$. Others can be obtained by cyclic transpositions of indices in K^k_{kij}. The values of components from this group in (4.10) are defined by the following formula

$$\begin{aligned} K^k_{kij} - \alpha^k_{kij} =& 2(\lambda_i - \lambda_j)\big[(\lambda_i - \lambda_k) + (\lambda_j - \lambda_k)\big] \\ & (\partial_i\lambda_k\partial_j\lambda_k - \partial_i\lambda_j\partial_j\lambda_k - \partial_j\lambda_i\partial_i\lambda_k) \end{aligned} \tag{4.11}$$

From (4.11) we see that the difference $K^k_{kij} - \alpha^k_{kij}$ contains only the derivatives of the first order. Now taking the tensor $\mathbf{N}$ we construct another tensor $\mathbf{M}$. Corresponding polylinear form is the following

$$\begin{aligned} &\mathbf{M}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{N}(\mathbf{X}, \mathbf{A}\mathbf{N}(\mathbf{Y}, \mathbf{Z})) + \\ &+ \mathbf{N}(\mathbf{A}\mathbf{X}, \mathbf{N}(\mathbf{Y}, \mathbf{Z})) - \mathbf{N}(\mathbf{N}(\mathbf{X}, \mathbf{Z}), \mathbf{A}\mathbf{Y}) + \\ &+ \mathbf{N}(\mathbf{N}(\mathbf{X}, \mathbf{Y}), \mathbf{A}\mathbf{Z}) - \mathbf{N}(\mathbf{X}, \mathbf{N}(\mathbf{A}\mathbf{Y}, \mathbf{Z})) - \\ &- \mathbf{N}(\mathbf{X}, \mathbf{N}(\mathbf{Y}, \mathbf{A}\mathbf{Z})) \end{aligned} \tag{4.12}$$

For the tensor $\mathbf{M}$ from (4.12) computed in the frame of eigenvectors of the operator $\mathbf{A}(\mathbf{u})$ we have

$$\mathbf{M}(\mathbf{X}_k, \mathbf{X}_i, \mathbf{X}_j) = M^k_{kij}\mathbf{X}_k + M^i_{kij}\mathbf{X}_i + M^j_{kij}\mathbf{X}_j \tag{4.13}$$

For the of components components of $\mathbf{M}$ in (4.13) we get

$$\begin{aligned} M^k_{kij} =& -(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)(\lambda_j - \lambda_k) \\ & (\partial_i\lambda_k\partial_j\lambda_k - \partial_i\lambda_j\partial_j\lambda_k - \partial_j\lambda_i\partial_i\lambda_k) \end{aligned} \tag{4.14}$$

The coefficients M^i_{kij} and M^j_{kij} aren't of interest for us now. Comparing (4.11) with (4.14) we define another tensor $\mathbf{Q}$

$$\begin{aligned} &\mathbf{Q}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{K}(\mathbf{A}\mathbf{X}, \mathbf{A}\mathbf{Y}, \mathbf{Z}) - \mathbf{K}(\mathbf{A}^2\mathbf{X}, \mathbf{Y}, \mathbf{Z}) - \\ &- \mathbf{K}(\mathbf{X}, \mathbf{A}\mathbf{Y}, \mathbf{A}\mathbf{Z}) + \mathbf{K}(\mathbf{A}\mathbf{X}, \mathbf{Y}, \mathbf{A}\mathbf{Z}) + 4\mathbf{M}(\mathbf{A}\mathbf{X}, \mathbf{Y}, \mathbf{Z}) - \\ &- 2\mathbf{M}(\mathbf{X}, \mathbf{A}\mathbf{Y}, \mathbf{Z}) - 2\mathbf{M}(\mathbf{X}, \mathbf{Y}, \mathbf{A}\mathbf{Z}) \end{aligned} \tag{4.15}$$

For the components of this tensor in the frame of eigenvectors of operator $\mathbf{A}(\mathbf{u})$ we get the relationship

$$\mathbf{Q}(\mathbf{X}_k, \mathbf{X}_i, \mathbf{X}_j) = Q^k_{kij}\mathbf{X}_k + Q^i_{kij}\mathbf{X}_i + Q^j_{kij}\mathbf{X}_j \tag{4.16}$$

which is analogous to (4.10) and (4.13). The components in (4.16) which are of interest for us are expressed through (4.7). They have the form

$$Q^k_{kij} = -(\lambda_i - \lambda_k)(\lambda_j - \lambda_k)\alpha^k_{kij} \tag{4.17}$$

Now on a base of (4.17) we are able to construct the semihamiltonity tensor which is the main goal of the whole paper. It is defined by the following formula

$$\begin{aligned}&\mathbf{P}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \mathbf{A}\mathbf{Q}(\mathbf{X}, \mathbf{A}\mathbf{Y}, \mathbf{Z}) + \\ &+ \mathbf{A}\mathbf{Q}(\mathbf{X}, \mathbf{Y}, \mathbf{A}\mathbf{Z}) - \mathbf{A}^2\mathbf{Q}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) - \mathbf{Q}(\mathbf{X}, \mathbf{A}\mathbf{Y}, \mathbf{A}\mathbf{Z})\end{aligned} \tag{4.18}$$

It is easy to check that for arbitrary three vectors from the frame $X^1, \ldots, X^n$ one has the relationship

$$\mathbf{P}(\mathbf{X}_k, \mathbf{X}_i, \mathbf{X}_j) = (\lambda_i - \lambda_k)^2(\lambda_j - \lambda_k)^2\alpha^k_{kij}\mathbf{X}_k \tag{4.19}$$

As a result we proved the following theorem (the invariant criterion of semihamiltonity).

Theorem 9. *The diagonalizable operator of general position* $\mathbf{A}$ *with mutually distinct eigenvalues is semi-Hamiltonian if and only if when associated tensor* $\mathbf{P}$ *from* (4.18) *is identically zero.*

The proof of this theorem follows directly from (4.8) and (4.9). It doesn't require any comments. Concluding all above considerations we give the formulae which enable us to calculate the components of the tensor of semihamiltonity $\mathbf{P}$ from the matrix $\mathbf{A}(\mathbf{u})$

$$\begin{aligned}P^s_{kij} = \sum_{p=1}^{n}\sum_{q=1}^{n}\bigl(&A^s_p Q^p_{kqj} A^q_i + A^s_p Q^p_{kiq} A^q_j - \\ &- A^s_q A^q_p Q^p_{kij} - Q^s_{kpq} A^p_i A^q_j\bigr)\end{aligned} \tag{4.20}$$

This formula is derived from (4.18). Components of the tensor $\mathbf{Q}$ in the formula (4.20) are calculated on a base of (4.15)

$$\begin{aligned}Q^s_{kij} = &\sum_{p=1}^{n}\sum_{q=1}^{n}\bigl(A^p_k K^s_{pqj} A^q_i + A^p_k K^s_{piq} A^q_j - A^p_q A^q_k K^s_{pij} - \\ &- K^s_{kpq} A^p_i A^q_j\bigr) + \sum_{p=1}^{n}\bigl(4A^p_k M^s_{pij} - 2M^s_{kpj} A^p_i - 2M^s_{kip} A^p_j\bigr)\end{aligned}$$

Components of $\mathbf{M}$ in the above formula are found from (4.12)

$$\begin{aligned}M^s_{kij} = &\sum_{p=1}^{n}\sum_{q=1}^{n}\bigl(N^s_{kp} A^p_q N^q_{ij} + N^s_{pq} A^p_k N^q_{ij} - \\ &- N^s_{pq} N^p_{ik} A^q_j - N^s_{pq} N^p_{kj} A^q_i - N^s_{kp} N^p_{iq} A^q_j - N^s_{kp} N^p_{qj} A^q_i\bigr)\end{aligned}$$

Tensor $\mathbf{K}$ are computed through tensor of Nijenhuis with the use of the bracket of Froelicher and Nijenhuis on a base of formula (4.9). Let's take $\mathbf{B} = \mathbf{A}^2$. Then for the components of $\mathbf{K}$ we have

$$K^s_{kij} = \sum_{p=1}^{n} \big(B^s_p \partial_k N^p_{ij} - B^p_k \partial_p N^s_{ij} + + N^p_{ij} \partial_p B^s_k - N^s_{kp} \partial_i B^p_j + N^s_{kp} \partial_j B^p_i \big) + \dots \tag{4.21}$$

By dots in (4.21) we denote 10 summand that can be obtained from 5 explicit summands by cyclic transposition of indices i, j and k.

The above formulae huge enough for direct calculations. However modern computer systems for analytical calculations solve this problem for any particular equations of hydrodynamical type in applications.

References

1. Dubrovin B.A., Novikov S.P., *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamilton's theory.*, Uspehi Mat. Nauk **44** (1989), no. 6, 29–98.
2. Tsarev S.P., *Geometry of hamiltonian systems of hydrodynamical type. Generalized method of hodograph.*, Izvestiya AN SSSR, ser. Metem. **54** (1990), no. 5, 1048–1068.
3. Haantjes A., *On X_{n-1}-forming sets of eigenvectors.*, Indagationes Mathematicae **17** (1955), no. 2, 158–162.
4. Ferapontov E.V., Tsarev S.P., *Systems of hydrodynamical type appearing in chromatography. Riemann invariants and exact solutions.*, Mathematical modeling **3** (1991), no. 2, 82–91.
5. Nijenhuis A., *X_{n-1}-forming sets of eigenvectors.*, Indagationes Mathematicae **13** (1951), no. 2, 200–212.
6. Froelicher A., Nijenhuis A., *Some new cohomology invariants for complex manifolds.*, Proc. Koninkl. nederl. akad. wetensch. **A59** (1956), no. 5, 540–564.
7. Berezin F.A., *Introduction into the algebra and analysis with anticommutating variables.*, Moscow State University publishers, Moscow, 1983.
8. Kobayashi Sh., Nomizu K., *Foundations of differential geometry. Vol. 1.*, Interscience Publishers, New York London, 1963.

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