ON THE POINT TRANSFORMATIONS FOR THE SECOND ORDER DIFFERENTIAL EQUATIONS. I.

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Abstract. Point transformations for the ordinary differential equations of the form $y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,(y')^2 + S(x,y)\,(y')^3$ are considered. Some classical results are resumed. Solution for the equivalence problem for the equations of general position is described.

1. Introduction.

Let's consider an ordinary differential equation of the second order with the right hand side being cubic polynomial in y':

$$y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,(y')^2 + S(x,y)\,(y')^3. \tag{1.1}$$

Class of the equations (1.1) is conserved under the point transformations of the form

$$\begin{cases} \tilde{x} = \tilde{x}(x,y), \\ \tilde{y} = \tilde{y}(x,y). \end{cases} \tag{1.2}$$

This means that after the change of variables (1.2) in any one of such equations we shall obtain another equation of the same form

$$\tilde{y}'' = \tilde{P}(\tilde{x},\tilde{y}) + 3\,\tilde{Q}(\tilde{x},\tilde{y})\,\tilde{y}' + 3\,\tilde{R}(\tilde{x},\tilde{y})\,(\tilde{y}')^2 + \tilde{S}(\tilde{x},\tilde{y})\,(\tilde{y}')^3. \tag{1.3}$$

If two particular equations (1.1) and (1.3) are fixed, the question on the existence of the point transformation (1.2) that transfer one of these equations into another is known as *the problem of equivalence*. In special case when the equation (1.3) is trivial $\tilde{y}'' = 0$ this problem was solved in classical papers by Tresse [1] and Cartan [2].

Another particular case for the equivalence problem is connected with Painleve equations. These are six equations of the form (1.1) with meromorphic coefficients defined by the condition that their common solution $y = y(x, c_1, c_2)$ considered as a function in x has no singularities except for the poles (see [3] and [4]). Painleve equations became very popular with the advent of the inverse scattering method,

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since they arise as self-similar solutions for various equations, which are integrable by this method. First two Painleve equations are the following

$$\tilde y'' = \tilde y^2 + \tilde x, \tag{1.4}$$

$$\tilde y'' = \tilde y^3 + \tilde x\,\tilde y + a, \tag{1.5}$$

where $a = \text{const}$. In [5] the problem of equivalence for the equations (1.4) and (1.5) was considered for the point transformations of the special form:

$$\begin{cases} \tilde x = \tilde x(x), \\ \tilde y = \tilde y(x, y). \end{cases} \tag{1.6}$$

Transformations (1.6) constitute a subset in the set of general point transformations (1.2). The generalization of the result from [5] for the case of arbitrary point transformations (1.2) was obtained in [6].

Main goal of our paper is to resume some classical constructions from [1] and [2], and apply them to the solution of the equivalence problem for the equations (1.1) being in general position. This is the most broad class of equations of the form (1.1), but nevertheless many famous equations appear to be out of this class. All six Painleve equations are not in general position, therefore the solution of the equivalence for them requires the separate consideration.

2. Point transformations.

Let's suppose the point transformation (1.2) to be regular. Denote by T and S direct and inverse matrices of Jacoby for the transformation (1.2)

$$S = \begin{Vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{Vmatrix}, \qquad T = \begin{Vmatrix} \tilde x_{1.0} & \tilde x_{0.1} \\ \tilde y_{1.0} & \tilde y_{0.1} \end{Vmatrix}. \tag{2.1}$$

By means of double indices in (2.1) and in what follows we indicate partial derivatives. For the function $f(u, v)$ $f_{p.q}$ we denote the result of differentiation p-times with respect to its first argument and q-times with respect to the second argument.

The formula for transforming the first order derivatives by the point transformations (1.2) has the following form:

$$y' = \frac{y_{1.0} + y_{0.1}\,\tilde y'}{x_{1.0} + x_{0.1}\,\tilde y'}. \tag{2.2}$$

Analogous formula for the second order derivatives is written as follows:

$$\begin{aligned} y'' = {} & \frac{(x_{1.0} + x_{0.1}\,\tilde y')(y_{2.0} + 2\,y_{1.1}\,\tilde y' + y_{0.2}\,(\tilde y')^2 + y_{0.1}\,\tilde y'')}{(x_{1.0} + x_{0.1}\,\tilde y')^3} - \\ & - \frac{(y_{1.0} + y_{0.1}\,\tilde y')(x_{2.0} + 2\,x_{1.1}\,\tilde y' + x_{0.2}\,(\tilde y')^2 + x_{0.1}\,\tilde y'')}{(x_{1.0} + x_{0.1}\,\tilde y')^3}. \end{aligned} \tag{2.3}$$

By substituting (2.2) and (2.3) into (1.1) we define the transformation rule for the coefficients of the equations (1.1) by the point transformation (1.2). In order to write this rule in a compact form let's construct a three dimensional array with the following components determined by the coefficients of the equation (1.1)

$$\begin{aligned} &\theta_{111} = P, && \theta_{112} = \theta_{121} = \theta_{211} = Q,\\ &\theta_{122} = \theta_{212} = \theta_{221} = R, && \theta_{222} = S. \end{aligned} \tag{2.4}$$

As we can see now from (2.4) the array θ_{ijk} is symmetric in each pair of indices. Let's raise one of these indices

$$\theta^k_{ij} = \sum_{r=1}^{2} d^{kr}\,\theta_{rij}, \tag{2.5}$$

by means of contraction with the following skew-symmetric matrix d^{ij}

$$d_{ij} = d^{ij} = \begin{Vmatrix} 0 & 1\\ -1 & 0 \end{Vmatrix}. \tag{2.6}$$

The transformation rule for the quantities (2.5) under the point change of variables (1.2) can be written as

$$\theta^k_{ij} = \sum_{m=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2} S^k_m\,T^p_i\,T^q_j\,\tilde\theta^m_{pq} + \sum_{m=1}^{2} S^k_m\,\frac{\partial T^m_i}{\partial x^j} - \frac{\tilde\sigma_i\,\delta^k_j + \tilde\sigma_j\,\delta^k_i}{3}, \tag{2.7}$$

where $x^1 = x$, $x^2 = y$, $\tilde x^1 = \tilde x$, $\tilde x^2 = \tilde y$ and where

$$\tilde\sigma_i = \frac{\partial \ln\det T}{\partial x^i}, \qquad \delta^k_i = \begin{cases} 1 & \text{for } i = k,\\ 0 & \text{for } i \neq k. \end{cases} \tag{2.8}$$

Because of the last summand with $\tilde\sigma_i$ and $\tilde\sigma_j$ the formula (2.7) differs from the standard rule of transformation for the components of connection (see [7]). But this shouldn't prevent us to construct the four dimensional array of quantities which in Riemannian geometry is known as a curvature tensor:

$$\Omega^k_{rij} = \frac{\partial\theta^k_{jr}}{\partial u^i} - \frac{\partial\theta^k_{ir}}{\partial u^j} + \sum_{q=1}^{2}\theta^k_{iq}\theta^q_{jr} - \sum_{q=1}^{2}\theta^k_{jq}\theta^q_{ir}. \tag{2.9}$$

Under the local change of variables (1.2) the quantities Ω^k_{rij} in (2.9) are transformed according to the rule

$$\Omega^k_{rij} = \sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{p=1}^{2}\sum_{q=1}^{2} S^k_m\,T^n_r\,T^p_i\,T^q_j\,\tilde\Omega^m_{npq} - \frac{\tilde\sigma_{ir}\,\delta^k_j - \tilde\sigma_{jr}\,\delta^k_i}{3}, \tag{2.10}$$

which is different from the rule of transformation for the components of tensor. The quantities $\tilde\sigma_{ij}$ in (2.10) are determined by $\tilde\sigma_i$ from (2.8) according to the formula

$$\tilde\sigma_{ij} = \frac{\partial\tilde\sigma_j}{\partial x^i} - \sum_{q=1}^{2}\theta^q_{ij}\,\tilde\sigma_q - \frac{1}{3}\,\tilde\sigma_i\,\tilde\sigma_j. \tag{2.11}$$

Now let's contract the array Ω^k_{rij} by the pair of indices k and i

$$\Omega_{rj}=\sum_{k=1}^{2}\Omega^k_{rkj}. \tag{2.12}$$

In Riemannian geometry the result of contraction (2.12) is known as the tensor of Ricci. But here we obtain the two-dimensional array Ω_{rj} which is not a tensor. Under the point transformations (1.2) the quantities Ω_{rj} are transformed as follows

$$\Omega_{rj}=\sum_{n=1}^{2}\sum_{q=1}^{2}T^n_r\,T^q_j\,\tilde\Omega_{nq}+\frac{1}{3}\,\tilde\sigma_{jr}. \tag{2.13}$$

Formula (2.13) is derived from (2.10). Here we deal with two dimensional manifold — coordinate plane (x,y). Because of this two-dimensionality all components of the array Ω^k_{rij} can be recovered from Ω_{rj}:

$$\Omega^k_{rij}=\Omega_{rj}\,\delta^k_i-\Omega_{ri}\,\delta^k_j. \tag{2.14}$$

Array of quantities Ω_{ij} is symmetric in i and j. Let's write down the values of its components in explicit form

$$\begin{aligned}
&\Omega_{12}=\Omega_{21}=R_{1.0}-Q_{0.1}+P\,S-Q\,R,\\
&\Omega_{11}=Q_{1.0}-P_{0.1}+2\,P\,R-2\,Q^2,\\
&\Omega_{22}=S_{1.0}-R_{0.1}+2\,S\,Q-2\,R^2.
\end{aligned} \tag{2.15}$$

Now we calculate derivatives $\nabla_i\Omega_{jk}$, using θ^k_{ij} as components of connection

$$\nabla_i\Omega_{jk}=\frac{\partial\Omega_{jk}}{\partial x^i}-\sum_{r=1}^{2}\theta^r_{ij}\,\Omega_{rk}-\sum_{r=1}^{2}\theta^r_{ik}\,\Omega_{jr}. \tag{2.16}$$

and by (2.16) we construct another three-dimensional array W_{ijk} skew-symmetric in first pair of indices

$$W_{ijk}=\nabla_i\Omega_{jk}-\nabla_j\Omega_{ik}. \tag{2.17}$$

Quantities (2.16) don't form a tensor, the use of the sign of covariant derivative ∇_i in (2.16) is quite formal. However the quantities (2.17) do form a tensor. This significant fact can be checked by direct calculations which are based on (2.10), (2.11) and (2.13).

Because of skew-symmetry in i and j the number of nonzero components of tensor W is 2. The most simple way to extract them is to contract W_{ijk} with the matrix d^{ij} defined by the formula (2.6)

$$\alpha_k=\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}W_{ijk}\,d^{ij}. \tag{2.18}$$

Quantities α_1 and α_2 can be calculated directly from the coefficients of the equation (1.1). Let's write down the appropriate formulas

$$\begin{aligned} A = \alpha_1 = P_{0.2} - 2\,Q_{1.1} + R_{2.0} + 2\,P\,S_{1.0} + S\,P_{1.0} - \\ - 3\,P\,R_{0.1} - 3\,R\,P_{0.1} - 3\,Q\,R_{1.0} + 6\,Q\,Q_{0.1}, \\ B = \alpha_2 = S_{2.0} - 2\,R_{1.1} + Q_{0.2} - 2\,S\,P_{0.1} - P\,S_{0.1} + \\ + 3\,S\,Q_{1.0} + 3\,Q\,S_{1.0} + 3\,R\,Q_{0.1} - 6\,R\,R_{1.0}. \end{aligned} \tag{2.19}$$

Components d^{ij} do not form a tensor, therefore the result of contraction (2.18) is not a tensor too. Arrays α_k and d^{ij} belong to special class geometrical objects which are known as *pseudotensors*.

Definition 2.1. Pseudotensorial field of the type (r, s) and weight m is an array of quantities $F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ which under the change of variables (1.2) transforms as follows

$$F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = (\det T)^m \sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots\,S^{i_r}_{p_r}\;T^{q_1}_{j_1}\ldots\,T^{q_s}_{j_s}\;\tilde F^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}. \tag{2.20}$$

Traditional tensorial fields can be treated as pseudotensorial fields of the weight $m = 0$ in (2.20). Quantities d^{ij} in (2.6) and quantities α_k in (2.18) form the components of pseudotensorial fields of the weight $m = 1$. As an important consequence of pseudotensorial character of the quantities A and B in (2.19) we get the following lemma.

Lemma 2.1. *If both parameters A and B are zero $A = B = 0$ for the initial equation* (1.1)*, then they both are zero $\tilde A = 0$ and $\tilde B = 0$ for the transformed equation* (1.3)*.*

Now let's calculate $\nabla_i\alpha_k$ using θ^k_{ij} from (2.5) as the components of connection. Then determine the quantities β_i according to the formula

$$\beta_i = 3\sum_{k=1}^{2}\sum_{r=1}^{2}\nabla_i\alpha_k\,d^{kr}\,\alpha_r + \sum_{k=1}^{2}\sum_{r=1}^{2}\nabla_r\alpha_k\,d^{kr}\,\alpha_i. \tag{2.21}$$

Quantities (2.21) appear to be the components of pseudocovectorial field of the weight 3. Raising indices in α_i and β_i we get two pseudovectorial fields with the weights 2 and 4 respectively

$$\alpha^i = \sum_{k=1}^{2} d^{ik}\,\alpha_k, \qquad\qquad \beta^i = \sum_{k=1}^{2} d^{ik}\,\beta_k, \tag{2.22}$$

We can calculate β^1 and β^2 directly from the coefficients of the equation (1.1):

$$\begin{aligned} \beta^1 = G = -B\,B_{1.0} - 3\,A\,B_{0.1} + 4\,B\,A_{0.1} + 3\,S\,A^2 - 6\,R\,B\,A + 3\,Q\,B^2, \\ \beta^2 = H = -A\,A_{0.1} - 3\,B\,A_{1.0} + 4\,A\,B_{1.0} - 3\,P\,B^2 + 6\,Q\,A\,B - 3\,R\,A^2. \end{aligned} \tag{2.23}$$

Now let's define the quantity F by means of contraction of the fields (2.18) and (2.22)

$$3\,F^5=\sum_{i=1}^{2}\alpha_i\,\beta^i=-\sum_{i=1}^{2}\beta_i\,\alpha^i=A\,G+B\,H. \tag{2.24}$$

We can also write down the explicit formula for F

$$\begin{aligned}F^5=A\,B\,A_{0.1}+B\,A\,B_{1.0}-A^2\,B_{0.1}-B^2\,A_{1.0}-\\-P\,B^3+3\,Q\,A\,B^2-3\,R\,A^2\,B+S\,A^3.\end{aligned} \tag{2.25}$$

The quantity F given by the formula (2.25) is a pseudoscalar field of the weight 1. As an immediate consequence of this fact we obtain the following lemma.

Lemma 2.2. *Vanishing of the parameter $F=0$ for initial equation* (1.1) *is equivalent to vanishing $\tilde F=0$ for the transformed equation* (1.3).

Lemma 2.2 separate two quite different cases in the study of the equations (1.1): $F=0$ and $F\neq 0$. The case $F\neq 0$ appears to be more structured from the geometrical point of view — it is the case of general position.

3. Case of general position.

Let $F\neq 0$. This means that pseudovectorial fields α and β are not collinear. We can form two vectorial fields $\mathbf{X}$ and $\mathbf{Y}$ from them

$$X^i=\frac{\alpha^i}{F^2},\qquad\qquad Y^i=\frac{\beta^i}{F^4}. \tag{3.1}$$

Nonzero pseudoscalar field F lets us to define the quantities

$$\varphi_i=-\frac{\partial\ln F}{\partial x^i}, \tag{3.2}$$

which under the point change of variables (1.2) should transform as follows

$$\varphi_i=\sum_{j=1}^{2}T_i^j\,\tilde\varphi_j-\sigma_i. \tag{3.3}$$

Compare this rule of transformation for the quantities (3.2) with the rule of transformation for θ_{ij}^k. Such comparison of (3.3) with (2.10) enables us to modify the quantities θ_{ij}^k converting them into the components of an affine connection

$$\varGamma_{ij}^k=\theta_{ij}^k-\frac{\varphi_i\,\delta_j^k+\varphi_j\,\delta_i^k}{3} \tag{3.4}$$

Noncollinear vector fields $\mathbf{X}$ and $\mathbf{Y}$ form the moving frame in the coordinate plane (x,y). Consider the components of the connection (3.4), related to this frame. They are defined as coefficients Γ_{11}^1 in the following expansions:

$$\begin{aligned}\nabla_{\mathbf{X}}\mathbf{X}&=\Gamma_{11}^1\,\mathbf{X}+\Gamma_{11}^2\,\mathbf{Y}, &\qquad \nabla_{\mathbf{X}}\mathbf{Y}&=\Gamma_{12}^1\,\mathbf{X}+\Gamma_{12}^2\,\mathbf{Y},\\ \nabla_{\mathbf{Y}}\mathbf{X}&=\Gamma_{21}^1\,\mathbf{X}+\Gamma_{21}^2\,\mathbf{Y}, &\qquad \nabla_{\mathbf{Y}}\mathbf{Y}&=\Gamma_{22}^1\,\mathbf{X}+\Gamma_{22}^2\,\mathbf{Y}.\end{aligned} \tag{3.5}$$

In contrast to the quantities $\varGamma^k_{ij}$ in (3.4) the coefficients Γ^k_{ij} in (3.5) do not change under the point transformations (1.2). They are scalar fields, i.e. they are *scalar invariants* of the equation (1.1). Let's numerate them as follows: $I_1 = \Gamma^1_{11}$, $I_2 = \Gamma^2_{11}$, $I_3 = \Gamma^1_{12}$, ... , $I_8 = \Gamma^2_{22}$. List of scalar invariants of the equation (1.1) can be continued. We can differentiate these invariants along the vector fields $\mathbf{X}$ and $\mathbf{Y}$ obtaining as a result more and more new invariants: $I_9 = \mathbf{X}I_1$, ... , $I_{16} = \mathbf{X}I_8$, $I_{17} = \mathbf{Y}I_1$, ... , $I_{24} = \mathbf{Y}I_8$. Repeating this procedure gives us 16 invariants in each step. The general structure of invariants distinguishes three different cases

(1) in the infinite sequence of invariants I_1, I_2, I_3, ... there is a pair of functionally independent ones;
(2) all invariant in the sequence I_1, I_2, I_3, ... are functionally dependent, but not all are identically constant;
(3) all invariants in the sequence I_1, I_2, I_3, ... are constants.

Case 1. Consider the set of pairs of invariants arranged in a lexicographic ordering. Then choose the first pair of functionally independent invariants in this ordering: (I_p, I_q). We can use $I_p(x, y)$ and $I_q(x, y)$ in order to define point transformation

$$\begin{cases} \tilde{x} = I_p(x, y), \\ \tilde{y} = I_q(x, y). \end{cases} \tag{3.6}$$

Variables $\tilde{x}$ and $\tilde{y}$ defined by invariants I_p and I_q are natural to be taken for the *canonical variables* of the equation (1.1). The equation (1.3) obtained as a result of transformation (3.6) is natural to call the *canonical form* of the equation (1.1). Solution of the equivalence problem in this case is given by the following obvious theorem.

Theorem 3.1. *Two equations of the form* (1.1) *with nonzero parameters* F *and with functionally independent invariants are equivalent if and only if they have the same canonical form.*

Before considering the second case let's write down explicit formulas for the first eight invariants I_1, I_2, ... , I_8. For I_1 and I_3 we have

$$\begin{aligned} I_1 = {} & \frac{B\,(G\,A_{1.0} + H\,B_{1.0})}{3\,F^7} - \frac{A\,(G\,A_{0.1} + H\,B_{0.1})}{3\,F^7} + \frac{4\,(A\,F_{0.1} - B\,F_{1.0})}{3\,F^3} + \\ & + \frac{G\,B^2\,P}{3\,F^7} + \frac{(H\,B^2 - 2\,G\,B\,A)\,Q}{3\,F^7} + \frac{(G\,A^2 - 2\,H\,B\,A)\,R}{3\,F^7} + \frac{H\,A^2\,S}{3\,F^7} \end{aligned} \tag{3.7}$$

$$\begin{aligned} I_3 = {} & \frac{B\,(H\,G_{1.0} - G\,H_{1.0})}{3\,F^9} - \frac{A\,(H\,G_{0.1} - G\,H_{0.1})}{3\,F^9} + \frac{H\,F_{0.1} + G\,F_{1.0}}{3\,F^5} + \\ & + \frac{B\,G^2\,P}{3\,F^9} - \frac{(A\,G^2 - 2\,H\,B\,G)\,Q}{3\,F^9} + \frac{(B\,H^2 - 2\,H\,A\,G)\,R}{3\,F^9} - \frac{A\,H^2\,S}{3\,F^9} \end{aligned} \tag{3.8}$$

Formula for I_7 has much more simple form than formulas (3.7) and (3.8)

$$\begin{aligned} I_7 = {} & \frac{G\,H\,G_{1.0} - G^2\,H_{1.0} + H^2\,G_{0.1} - H\,G\,H_{0.1}}{3\,F^{11}} + \\ & + \frac{G^3\,P + 3\,G^2\,H\,Q + 3\,G\,H^2\,R + H^3\,S}{3\,F^{11}} \end{aligned} \tag{3.9}$$

Invariant I_2 require no calculations — it is simply an identical constant

$$I_2=\frac{1}{3}. \tag{3.10}$$

In order to calculate rest four invariants let's define the following two quantities which are also scalar invariants

$$K=I_6-I_4=\Gamma^2_{21}-\Gamma^2_{12}=\frac{1}{F}\frac{\partial}{\partial x}\left(\frac{B}{F}\right)-\frac{1}{F}\frac{\partial}{\partial y}\left(\frac{A}{F}\right), \tag{3.11}$$

$$L=I_3-I_5=\Gamma^1_{12}-\Gamma^1_{21}=\frac{1}{F}\frac{\partial}{\partial x}\left(\frac{G}{F^3}\right)+\frac{1}{F}\frac{\partial}{\partial y}\left(\frac{H}{F^3}\right). \tag{3.12}$$

Then invariants I_4, I_5, I_6 and I_8 are defined by the following relationships

$$I_4=\Gamma^2_{12}=-I_1, \qquad I_5=\Gamma^1_{21}=I_3-L, \tag{3.13}$$

$$I_6=\Gamma^2_{21}=-I_1+K, \qquad I_8=\Gamma^2_{22}=-I_5. \tag{3.14}$$

The quantities K and L in (3.11) and (3.12) have clear geometrical interpretation due to the identity $[\mathbf{X},\,\mathbf{Y}]=\nabla_{\mathbf{X}}\mathbf{Y}-\nabla_{\mathbf{Y}}\mathbf{X}$ (see for instance [8]):

$$[\mathbf{X},\,\mathbf{Y}]=L\,\mathbf{X}-K\,\mathbf{Y}. \tag{3.15}$$

Choose the functions u and v so that commutator of the following vector fields

$$\tilde{\mathbf{X}}=\frac{\mathbf{X}}{u}, \qquad \tilde{\mathbf{Y}}=\frac{\mathbf{Y}}{v}, \tag{3.16}$$

be equal to zero. Because of (3.15) this condition gives two differential equations for the functions u and v in (3.16)

$$\mathbf{Y}u=-L\,u, \qquad \mathbf{X}v=-K\,v. \tag{3.17}$$

Written in coordinates the equations (3.17) appears to be linear differential equations of the first order. Such equations are solved by means of the method of characteristics (see [9]). Thus we can choose pair of nonzero functions u and v that satisfy the condition $[\tilde{\mathbf{X}},\,\tilde{\mathbf{Y}}]=0$ for the vector fields (3.16).

Any pair of commutating vector fields on a plane defines some curvilinear system of coordinates $(\tilde{x},\tilde{y})$ for which $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ form the coordinate frame. Let's perform the point transformation to the coordinates $\tilde{x}$ and $\tilde{y}$ defined by $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$. Here

$$\begin{aligned} &\tilde{\alpha}^1=\tilde{B}=u\,\tilde{F}^2, &&\tilde{\alpha}^2=-\tilde{A}=0,\\ &\tilde{\beta}^1=\tilde{G}=0, &&\tilde{\beta}^2=\tilde{H}=v\,\tilde{F}^4, \end{aligned} \tag{3.18}$$

Substituting $\tilde{A}=0$ and $\tilde{G}=0$ from (3.18) into (2.23) we get the following equations:

$$3\,\tilde{Q}\,\tilde{B}^2-\tilde{B}\,\tilde{B}_{1.0}=0, \qquad \tilde{H}=-3\,\tilde{P}\,\tilde{B}^2. \tag{3.19}$$

Taking into account (2.24) and the relationships $\tilde{B} = u\,\tilde{F}^2$ and $\tilde{H} = v\,\tilde{F}^4$ we can express $\tilde{P}$, $\tilde{Q}$ and the functions u and v via $\tilde{B}$ and $\tilde{F}$

$$u = \frac{\tilde{B}}{\tilde{F}^2}, \qquad v = \frac{3\,\tilde{F}}{\tilde{B}}, \qquad \tilde{P} = -\frac{\tilde{F}^5}{\tilde{B}^3}, \qquad \tilde{Q} = \frac{\tilde{B}_{1.0}}{3\,\tilde{B}}. \tag{3.20}$$

Take nonzero parameters $\tilde{B} \neq 0$ and $\tilde{F} \neq 0$ for the basic ones in order to express invariants I_1, I_3, I_7, K and L in terms of them

$$I_1 = \frac{4}{3}\,\frac{\tilde{F}\,\tilde{B}_{1.0} - \tilde{B}\,\tilde{F}_{1.0}}{\tilde{F}^3}, \qquad I_3 = \frac{\tilde{F}_{0.1} + 3\,\tilde{F}\,\tilde{R}}{\tilde{B}}, \qquad I_7 = \frac{9\,\tilde{F}^4\,\tilde{S}}{\tilde{B}^3}. \tag{3.21}$$

For the invariants K and L that determine the commutator of $\mathbf{X}$ and $\mathbf{Y}$ we get the following expressions

$$K = \frac{\tilde{F}\,\tilde{B}_{1.0} - \tilde{B}\,\tilde{F}_{1.0}}{\tilde{F}^3}, \qquad L = \frac{6\,\tilde{B}\tilde{F}_{0.1} - 3\,\tilde{F}\,\tilde{B}_{0.1}}{\tilde{B}^2}. \tag{3.22}$$

Invariants I_4, I_5, I_6, ${}_8$ are determined by the relationships (3.13) and (3.14). In addition to these relationships from comparison of (3.21) and (3.22) we derive

$$I_1 = \frac{4}{3}\,K. \tag{3.23}$$

Note that the relationship (3.23) can be derived directly from (3.7) and (3.11). However this require more complicated calculations.

Case 2. In this case all invariants I_k are functionally dependent but not all of them are constants. Find first nonconstant invariant $I = I_p$ among them. Then all invariants I_k can be expressed as some functions of I, i.e. $I_k = I_k(I)$. The same is true for invariants K and L in (3.17). We shall seek the solutions for the equations (3.17) in form of $u = u(I)$ and $v = v(I)$

$$u'(I)\,\mathbf{Y}I = -L(I)\,u(I), \qquad v'(I)\,\mathbf{X}I = -K(I)\,v(I). \tag{3.24}$$

Both derivatives of I along the vector fields $\mathbf{X}$ and $\mathbf{Y}$ are also the invariants in the sequence $I_1, I_2, I_3, \ldots$, therefore $\mathbf{X}I = \xi(I)$ and $\mathbf{Y}I = \zeta(I)$. Substituting this into (3.7) we bring the equations (3.24) to the form of ordinary differential equations for the functions $u(I)$ and $v(I)$

$$u'\,\zeta(I) = -L(I)\,u, \qquad v'\,\xi(I) = -K(I)\,v. \tag{3.25}$$

For $\xi(I) \neq 0$ and $\zeta(I) \neq 0$ the equations (3.25) are obviously solvable. Their solutions can be chosen nonzero $u(I) \neq 0$ and $v(I) \neq 0$. Because of $I \neq \text{const}$ and because of linear independence of the vectors $\mathbf{X}$ and $\mathbf{Y}$ the functions $\mathbf{X}I = \xi(I)$ and $\mathbf{Y}I = \zeta(I)$ cannot vanish simultaneously. Let's differentiate the invariant I along the commutator $[\mathbf{X}, \mathbf{Y}]$ of the vector fields $\mathbf{X}$ and $\mathbf{Y}$

$$[\mathbf{X}, \mathbf{Y}]I = \mathbf{X}(\mathbf{Y}I) - \mathbf{Y}(\mathbf{X}I) = \zeta'(I)\,\mathbf{X}I - \xi'(I)\,\mathbf{Y}I = \zeta'(I)\,\xi(I) - \xi'(I)\,\zeta(I).$$

On the other hand due to (3.15) for the same expression $[\mathbf{X},\,\mathbf{Y}]I$ we get

$$[\mathbf{X},\,\mathbf{Y}]I=L(I)\,\mathbf{X}I-K\,\mathbf{Y}I=L(I)\,\xi(I)-K(I)\,\zeta(I).$$

Combining these two relationships for $[\mathbf{X},\,\mathbf{Y}]I$ we derive the following equality

$$\zeta'(I)\,\xi(I)-\xi'(I)\,\zeta(I)=L(I)\,\xi(I)-K(I)\,\zeta(I). \tag{3.26}$$

On the base of (3.26) it's easy to find that $\xi(I)=0$ leads to $K(I)=0$, and $\zeta(I)=0$ leads to $L(I)=0$. Therefore if $\xi(I)=0$, we can take $v(I)=1$, and if $\zeta(I)=0$, we can take $u(I)=1$. This let's us satisfy the equations (3.24) and (3.25) in any case.

The pair of commutating vector fields (3.16) defined by the choice of functions $u(I)$ and $v(I)$ determines the choice of curvilinear coordinates $\tilde{x}$ and $\tilde{y}$ on the plane such that $\tilde{A}=0$ and $\tilde{G}=0$. For the parameters $\tilde{F}$ and $\tilde{B}$ from (3.20) we get

$$\tilde{F}=3\,u^{-1}\,v^{-1},\qquad\qquad \tilde{B}=9\,u^{-1}\,v^{-2}. \tag{3.27}$$

From (3.27) we see that $\tilde{F}=\tilde{F}(I)$ and $\tilde{B}=\tilde{B}(I)$. For the parameters (3.2) this gives

$$\varphi_1=-\frac{F'(I)\,\xi(I)}{u(I)\,F(I)},\qquad\qquad \varphi_2=-\frac{F'(I)\,\zeta(I)}{v(I)\,F(I)}. \tag{3.28}$$

Hence parameters φ_i are also functions in I. In the coordinates $\tilde{x}$ and $\tilde{y}$ the components of connection (3.4) can be expressed via the coefficients of the expansions (3.5) which are the scalar invariants

$$\begin{aligned}
&\tilde{\Gamma}^1_{11}=u^{-1}\,\Gamma^1_{11}-\xi\,u^{-2}\,u', &&\tilde{\Gamma}^2_{11}=v\,u^{-2}\,\Gamma^2_{11},\\
&\tilde{\Gamma}^1_{12}=\tilde{\Gamma}^1_{21}=v^{-1}\,\Gamma^1_{12}, &&\tilde{\Gamma}^2_{12}=\tilde{\Gamma}^2_{21}=u^{-1}\,\Gamma^2_{21},\\
&\tilde{\Gamma}^1_{22}=u\,v^{-2}\,\Gamma^1_{22}, &&\tilde{\Gamma}^2_{22}=v^{-1}\,\Gamma^2_{22}-\zeta\,v^{-2}\,v'.
\end{aligned} \tag{3.29}$$

In order to transfer from $\tilde{\Gamma}^k_{ij}$ to $\tilde{\theta}_{kij}$ we should perform an operation inverse to the raising of index in (2.5). This is done by means of the matrix $-d_{kp}$

$$\tilde{\theta}_{kij}=-\sum_{p=1}^{2}d_{kp}\,\tilde{\Gamma}^p_{ij}-\frac{\tilde{\varphi}_i\,d_{kj}+\tilde{\varphi}_j\,d_{ki}}{3}. \tag{3.30}$$

But the quantities $\tilde{\theta}_{kij}$ coincide with the coefficients of the equation (1.3) in coordinates $\tilde{x}$ and $\tilde{y}$. Therefore from (3.28), (3.29) and (3.30) we obtain

$$\tilde{P}=\tilde{P}(I),\qquad \tilde{Q}=\tilde{Q}(I),\qquad \tilde{R}=\tilde{R}(I),\qquad \tilde{S}=\tilde{S}(I). \tag{3.31}$$

Conclusion: coefficients of the transformed equation (1.3) and its parameters $\tilde{B}$ are $\tilde{F}$ the functions in I. Invariant I in turn is a function in $\tilde{x}$ and $\tilde{y}$. Let's study the dependence of I on $\tilde{x}$ and $\tilde{y}$. In order to do it let's consider the derivatives

$$I_{1.0}=\frac{\mathbf{X}I}{u}=\frac{\xi(I)}{u(I)}=h(I),\qquad\qquad I_{0.1}=\frac{\mathbf{Y}I}{v}=\frac{\zeta(I)}{v(I)}=k(I). \tag{3.32}$$

and calculate the second order derivative $I_{1.1}$ from (3.32). This can be done in two ways, therefore we get the relationship

$$h'(I)\,k(I) - k'(I)\,h(I) = 0, \tag{3.33}$$

which should be considered as the compatibility condition for the equations (3.32). The relationship (3.33) can be integrated. This gives the linear dependence of the functions $h(I)$ and $k(I)$, i.e. there are two constants C_1 and C_2 such that

$$C_2\,h(I) - C_1\,k(I) = 0. \tag{3.34}$$

The relationship (3.34) is the equation for the function I

$$C_2\,\frac{\partial I}{\partial \tilde{x}} - C_1\,\frac{\partial I}{\partial \tilde{y}} = 0. \tag{3.35}$$

The equation (3.35) is easily integrable by means of the method of characteristics. Denote $\tau = C_1\,\tilde{x} + C_2\,\tilde{y}$. Then the general solution of the differential equation (3.35) is given by an arbitrary function of one variable τ

$$I = I(\tau) = I(C_1\,\tilde{x} + C_2\,\tilde{y}). \tag{3.36}$$

Note that the functions $u(I)$ and $v(I)$ are determined by the differential equations (3.25) only up to a constant factor. This let's us make the constants C_1 and C_2 in (3.36) equal to unity if their initial values are not zero. Therefore the general form of dependence of τ on $\tilde{x}$ and $\tilde{y}$ can be reduced to the following special cases

$$\tau = \tilde{x} + \tilde{y}, \qquad \tau = \tilde{x}, \qquad \tau = \tilde{y}. \tag{3.38}$$

In any of these three cases defined by (3.38), we can start by choosing two arbitrary nonzero functions $\tilde{F}(\tau)$ and $\tilde{B}(\tau)$. Then define the coefficients $\tilde{P}(\tau)$ and $\tilde{Q}(\tau)$ for the equation (1.3) by means of formulas (3.20). And finally define the rest two coefficients $\tilde{R}(\tau)$ and $\tilde{S}(\tau)$ for the equation (1.3) by solving the system of ordinary differential equations derived from (2.19). For the case $\tau = \tilde{x} + \tilde{y}$ this system is as follows

$$\begin{aligned} &P'' - 2\,Q'' + R'' + 2\,P\,S' + (S - 3\,R)\,P' - 3\,(P + Q)\,R' + 6\,Q\,Q' = 0,\\ &S'' - 2\,R'' + Q'' - 2\,S\,P' - (P - 3\,Q)\,S' + 3\,(S + R)\,Q' - 6\,R\,R' = B. \end{aligned} \tag{3.39}$$

When $\tau = \tilde{x}$ the system of equations (3.39) should be replaced by the following one

$$\begin{aligned} &R'' + 2\,P\,S' + S\,P' - 3\,Q\,R' = 0,\\ &S'' + 3\,Q\,S' + 3\,S\,Q' - 6\,R\,R' = B. \end{aligned} \tag{3.40}$$

In the last third case $\tau = \tilde{y}$ from (3.20) we have $\tilde{Q} = 0$. Therefore the system of equations for $\tilde{R}$ and $\tilde{S}$ here is even simpler than (3.40)

$$\begin{aligned} &P'' + S\,P' - 3\,S\,R' - 3\,R\,P' = 0,\\ &-2\,S\,P' - P\,S' = B. \end{aligned} \tag{3.41}$$

The above procedure of choosing the coefficients of the equation (1.3) based on the equations (3.39), (3.40) and (3.41) gives the complete description of the canonical form of the equations (1.1) for the case of functionally dependent invariants.

Case 3. Remember that in this case all invariant in the sequence I_1, I_2, I_3, ... are identical constants. Indeed we can check that first eight of them are constants. Then all other invariants I_9, I_{10}, ... will be zero. In this case we can also construct the commutating vector fields (3.16) and choose curvilinear coordinates $\tilde{x}$ and $\tilde{y}$ defined by them. Functions u and v are the solutions of the equations (3.17) where now $K = \text{const}$ and $L = \text{const}$. These equations admit some arbitrariness in the choice of their solutions. We shall use this arbitrariness in order to make the equation (1.3) as simple as possible.

Let's show that invariants K and L cannot vanish simultaneously. If $K = L = 0$, then we can choose $u = v = 1$ and for the parameters $\tilde{F}$, $\tilde{B}$, $\tilde{P}$ and $\tilde{Q}$ from (3.20) we derive

$$\tilde{F} = 3, \qquad \tilde{B} = 9, \qquad \tilde{P} = -\frac{1}{3}, \qquad \tilde{Q} = 0. \tag{3.42}$$

Substituting (3.42) in (3.21) we can express $\tilde{R}$ and $\tilde{S}$ through invariants I_3 and I_7:

$$\tilde{R} = I_3, \qquad \tilde{S} = I_7. \tag{3.43}$$

Because of (3.42) and (3.43) and because of constancy of invariants $I_1, \ldots, I_8$ all coefficients in (1.3) should be the constants. Substituting them in (2.19) we get $\tilde{B} = 0$. This contradict to the equality $\tilde{B} = 9$ from (3.42).

First case is $K = 0$ and $L \neq 0$. Here we choose $v = 1$. The choice of u we implement in two steps. First we choose an arbitrary solution for the equation $\mathbf{Y}u = -L\,u$ from (3.17). Denote this preliminary choice by $\hat{u}$. It defines the curvilinear coordinates in which the differential equation $\mathbf{Y}u = -L\,u$ for u has the form:

$$\frac{\partial u}{\partial \hat{y}} = -L\,u. \tag{3.44}$$

Being the solution of the equation (3.44) the function $\hat{u}$ has the form $\hat{u}(\hat{x}, \hat{y}) = \hat{u}(\hat{x})\,e^{-L\hat{y}}$. Now we take one more solution for the equation (3.44) given by the formula $u(\hat{x}, \hat{y}) = e^{-L\hat{y}}$. This ultimate choice of functions $u = e^{-L\hat{y}}$ and $v = 1$ determines new coordinates $\tilde{x} = f(\hat{x})$ and $\tilde{y} = \hat{y}$ in which the function u has the form $u(\tilde{x}, \tilde{y}) = e^{-L\tilde{y}}$ from the very beginning. In these canonical coordinates from (3.20) we derive

$$\tilde{F} = 3\,e^{L\tilde{y}}, \qquad \tilde{B} = 9\,e^{L\tilde{y}}, \qquad \tilde{P} = -\frac{1}{3}\,e^{2L\tilde{y}}, \qquad \tilde{Q} = 0. \tag{3.45}$$

Then on the base of (3.21) we express $\tilde{R}$ and $\tilde{S}$ through the invariants I_3 and I_7

$$\tilde{R} = I_3 - \frac{L}{3}, \qquad \tilde{S} = I_7\,e^{-L\tilde{y}}. \tag{3.46}$$

By substituting (3.45) and (3.46) into (2.19) we find $I_3 = L$ and $I_7 = 9/L$. Therefore the equation (1.3) is written as

$$\tilde{y}'' = -\frac{1}{3}\, e^{2L\tilde{y}} + 2\,L\,(\tilde{y}')^2 + \frac{9}{L}\, e^{-L\tilde{y}}\,(\tilde{y}')^3. \tag{3.47}$$

The equation (3.47) is a canonical form for the equation (1.1) with identically constant invariants when $F \neq 0$, $K = 0$ and $L \neq 0$.

Now we consider another case $L = 0$ and $K \neq 0$. Take $u = 1$ and choose the function v satisfying the equation $\mathbf{X}v = -K\,v$ from (3.17) such that in canonical coordinates $\tilde{x}$ and $\tilde{y}$ it has the form $v = e^{-K\tilde{x}}$. We omit the details of such choice since it's quite similar to the choice of u in previous case. From (3.20) we determine the parameters $\tilde{F}$, $\tilde{B}$, $\tilde{P}$ and $\tilde{Q}$ in canonical coordinates

$$\tilde{F} = 3\,e^{K\tilde{x}}, \qquad \tilde{B} = 9\,e^{2K\tilde{x}}, \qquad \tilde{P} = -\frac{1}{3}\,e^{-K\tilde{x}}, \qquad \tilde{Q} = \frac{2}{3}\,K. \tag{3.48}$$

Coefficients $\tilde{R}$ and $\tilde{S}$ are expressed through I_3 and I_7 by means of (3.21)

$$\tilde{R} = I_3\,e^{K\tilde{x}}, \qquad \tilde{S} = I_7\,e^{2K\tilde{x}}. \tag{3.49}$$

By substituting (3.48) and (3.49) into the relationships (2.19) and taking into account $\tilde{A} = 0$ we get

$$\begin{aligned} &I_7 + K\,I_3 = 0,\\ &-8\,I_7 * K^2 + 6\,K * I_3^2 + 9 = 0. \end{aligned} \tag{3.50}$$

Canonical form of the equation (1.1) in this case is as follows

$$\tilde{y}'' = -\frac{1}{3}\,e^{-K\tilde{x}} + 2\,K\,\tilde{y}' + 3\,I_3\,e^{K\tilde{x}}\,(\tilde{y}')^2 + I_7\,e^{2K\tilde{x}}\,(\tilde{y}')^3. \tag{3.51}$$

Here parameters I_3 and I_7 are defined by their parameter K from the equations (3.50).

The rest case is $L \neq 0$ and $K \neq 0$. In this case we should the functions u and v simultaneously. First we choose two arbitrary functions satisfying the equations (3.17). They define curvilinear coordinates $\hat{x}$ and $\hat{y}$ in which the equations (3.17) have the following form

$$v\,\frac{\partial u}{\partial \hat{y}} = -L\,u, \qquad u\,\frac{\partial v}{\partial \hat{x}} = -K\,v. \tag{3.52}$$

General solution for the system of equations (3.52) is determined by two arbitrary functions in one variable $\hat{p}(\hat{x})$ and $\hat{q}(\hat{y})$

$$\hat{u} = -K\,\frac{\hat{p}(\hat{x}) + \hat{q}(\hat{y})}{\hat{p}'(\hat{x})}, \qquad \hat{v} = -L\,\frac{\hat{p}(\hat{x}) + \hat{q}(\hat{y})}{\hat{q}'(\hat{y})}. \tag{3.53}$$

The following special point transformation $\hat x = f(\tilde x)$ and $\hat y = g(\tilde y)$ transfer (3.53) into the solution of the equations analogous to (3.52) in new variables. Under this change of variable the functions $\hat p$ and $\hat q$ are transformed according to the rule

$$\tilde p(\tilde x) = \hat p(f(\tilde y)), \qquad \tilde q(\tilde x) = \hat q(g(\tilde y)). \tag{3.54}$$

The rule of transformation (3.54) let's us choose the solutions of the equations (3.17) so that in the appropriate variables $\tilde x$ and $\tilde y$ they are

$$\hat u = -K\,(\tilde x + \tilde y), \qquad \hat v = -L\,(\tilde x + \tilde y), \tag{3.55}$$

i.e. $\tilde p(\tilde x) = \tilde x$ and $\tilde q(\tilde y) = \tilde y$. Now we are only to substitute (3.55) into the formulas (3.20). For $\tilde F$ and $\tilde B$ such substitution gives

$$\tilde F = \frac{3}{K\,L\,(\tilde x + \tilde y)^2}, \qquad \tilde B = -\frac{9}{K\,L^2\,(\tilde x + \tilde y)^3}. \tag{3.56}$$

Further we should find $\tilde P$ and $\tilde S$. This is also done by means of relationships (3.20)

$$\tilde P = -\frac{L}{3\,K^2\,(\tilde x + \tilde y)}, \qquad \tilde Q = -\frac{1}{\tilde x + \tilde y}. \tag{3.57}$$

In order to calculate $\tilde R$ and $\tilde S$ we shall use the formulas (3.21). From them we derive

$$\tilde R = -\frac{-3\,I_3 - 2}{3\,(\tilde x + \tilde y)}, \qquad \tilde S = -\frac{I_7\,K}{L^2(\tilde x + \tilde y)}. \tag{3.58}$$

By substituting (3.57) and (3.58) into (2.19) we find the relationships that bind I_3 and I_7 with K and L

$$\begin{aligned} &3\,K\,I_7 + 3\,(K^2 - 2\,L)\,I_3 + 6\,L^2 - 8\,L\,K^2 = 0,\\ &6\,K\,I_3^2 - 7\,K\,L\,I_3 - 8\,K^2\,I_7 - I_7\,L + 9 = 0. \end{aligned} \tag{3.59}$$

Formulas (3.57), (3.58) and (3.58) completely determine the canonical form of the equation (1.1) with identically constant invariants when $F \neq 0$, $K \neq 0$ and $L \neq 0$.

4. Final remarks and acknowledgments.

Problem of equivalence considered in this paper has a long history started from the last century (see [10]). Here are some references to the papers concerning this problem. References [11] and [12] are the papers by E. Cartan. Papers [2], [11] and [12] are translated into Russian and published in the book [13]. We are grateful to E.G. Neufeld, who gave us to read this book. References [6], [14–17] are communicated us by V.V. Sokolov and V.E. Adler.

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