ON THE POINT TRANSFORMATIONS FOR THE EQUATION $y'' = P + 3\,Q\,y' + 3\,R\,{y'}^2 + S\,{y'}^3$.

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Abstract. For the equations $y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,{y'}^2 + S(x,y)\,{y'}^3$ the problem of equivalence is considered. Some classical results are resumed in order to prepare the background for the study of special subclass of such equations, which arises in the theory of dynamical systems admitting the normal shift.

1. Introduction.

Let's consider an ordinary differential equation of the second order with polynomial in y' right hand side of the following form:

$$y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,(y')^2 + S(x,y)\,(y')^3. \tag{1.1}$$

Class of the equations (1.1) is invariant under the point transformations

$$\begin{cases} \tilde{x} = \tilde{x}(x,y), \\ \tilde{y} = \tilde{y}(x,y). \end{cases} \tag{1.2}$$

This means that after implementing the change of variables (1.2) in such equation we obtain another equation of the same form:

$$\tilde{y}'' = \tilde{P}(\tilde{x},\tilde{y}) + 3\,\tilde{Q}(\tilde{x},\tilde{y})\,\tilde{y}' + 3\,\tilde{R}(\tilde{x},\tilde{y})\,(\tilde{y}')^2 + \tilde{S}(\tilde{x},\tilde{y})\,(\tilde{y}')^3. \tag{1.3}$$

Suppose that two particular equations of the form (1.1) are taken. The question on the existence of the point transformation (1.2) that transfer one of them into another is known as the problem of equivalence. The study of this problem has the long history (see [1–22]). This paper is aimed to sum up all results concerning the problem of equivalence for the equations (1.1), which are known to us, and prepare the background for the study of special subclass of such equations:

$$y'' = \Phi_y(x,y)\,(y'+1) + \Phi_x(x,y).$$

These equations arise in the theory of dynamical systems admitting the normal shift (see paper [23] for details).

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2. Point transformations and point symmetries.

Let's treat the pair of variables (x, y) in (1.1) as the coordinates of some point on a plane. Then the change of variables (1.2) can be treated as the change of one (curvilinear) system of coordinates for another. By such treatment the equation (1.1) defines some geometrical structure on the plane. On fixing some local coordinates this structure is expressed by four functions $P(x, y)$, $Q(x, y)$, $R(x, y)$ and $S(x, y)$.

We shall take the point transformation (1.2) to be regular. Denote by S and T direct and inverse transition matrices for the point transformation (1.2);

$$S = \begin{Vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{Vmatrix}, \qquad T = \begin{Vmatrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{Vmatrix}. \tag{2.1}$$

By means of double indices in (2.1) here and in what follows we denote partial derivatives, e. g. for the function $f(u, v)$ by $f_{p.q}$ we denote the derivative of p-th order in the first argument u, and of q-th order in the second argument v.

Formula for transforming the first derivatives by the point transformation (1.2) has the following form:

$$y' = \frac{y_{1.0} + y_{0.1}\,\tilde{y}'}{x_{1.0} + x_{0.1}\,\tilde{y}'}. \tag{2.2}$$

Analogous formula is available for transforming the second order derivatives:

$$\begin{aligned} y'' = {} & \frac{(x_{1.0} + x_{0.1}\,\tilde{y}')(y_{2.0} + 2\,y_{1.1}\,\tilde{y}' + y_{0.2}\,(\tilde{y}')^2 + y_{0.1}\,\tilde{y}'')}{(x_{1.0} + x_{0.1}\,\tilde{y}')^3} - \\ & - \frac{(y_{1.0} + y_{0.1}\,\tilde{y}')(x_{2.0} + 2\,x_{1.1}\,\tilde{y}' + x_{0.2}\,(\tilde{y}')^2 + x_{0.1}\,\tilde{y}'')}{(x_{1.0} + x_{0.1}\,\tilde{y}')^3}. \end{aligned} \tag{2.3}$$

By substituting (2.2) and (2.3) into (1.1), we derive the transformation rules for the coefficients of the equation (1.1). In order to express these rules in compact form, we introduce 3-dimensional array with the following components:

$$\begin{aligned} &\theta_{111} = P, && \theta_{112} = \theta_{121} = \theta_{211} = Q, \\ &\theta_{122} = \theta_{212} = \theta_{221} = R, && \theta_{222} = S. \end{aligned} \tag{2.4}$$

As we can see from (2.4), the array θ_{ijk} is symmetric with respect to any pair of indices. Let's raise one of these indices

$$\theta^k_{ij} = \sum_{r=1}^{2} d^{kr}\,\theta_{rij} \tag{2.5}$$

by means of skew-symmetric matrix d^{ij} with the following components:

$$d_{ij} = d^{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{2.6}$$

We are able to write the transformation rule for the quantities θ^k_{ij} defined in (2.5). By the change of variables (1.2) they are transformed as follows:

$$\theta^k_{ij}=\sum^2_{m=1}\sum^2_{p=1}\sum^2_{q=1}S^k_m\,T^p_i\,T^q_j\,\tilde{\theta}^m_{pq}+\sum^2_{m=1}S^k_m\,\frac{\partial T^m_i}{\partial x^j}-\frac{\tilde{\sigma}_i\,\delta^k_j+\tilde{\sigma}_j\,\delta^k_i}{3},\tag{2.7}$$

where $x^1=x$, $x^2=y$, $\tilde{x}^1=\tilde{x}$, $\tilde{x}^2=\tilde{y}$, and where we used the following notations:

$$\tilde{\sigma}_i=\frac{\partial\ln\det T}{\partial x^i},\qquad\qquad \delta^k_i=\begin{cases}1 & \text{for } i=k,\\ 0 & \text{for } i\neq k.\end{cases}\tag{2.8}$$

Last summand with the quantities $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ differs the formula (2.7) from the standard transformation rule for the components of affine connection (see [24]). This is the very circumstance that determines the complexity of the structures connected with the equation (1.1). It prevent us from direct use of standard methods of differential geometry.

Apart from the treatment just described, one has another treatment for the change of variables (1.2). Taking local coordinates on the plane for fixed, we can treat (1.2) as a map f which transform the point with coordinates (x,y) into the point with coordinates $(\tilde{x},\tilde{y})$. Each solution of the equation (1.1) is the function $y(x)$, its graph is some curve on the plane. The transformation f maps this curve onto another curve, which also can be considered as the graph of some function.

Definition 2.1. The transformation of the plane f is called *the point symmetry* of the equation (1.1) if it maps each solution of this equation into another solution of the same equation.

The rules of transformation for the coefficients of the equation (1.1) under the mapping of the plane f are determined by the same formula (2.7). However the Jacobi matrices T and S are now interpreted as differentials of direct and inverse maps f_* and f_*^{-1}. According to the definition 2.1, the map f is the point symmetry for the equation (1.1) if it preserves the geometrical structure connected with this equation. If f is the point symmetry for some equation (1.1), then f^{-1} is also the point symmetry for the same equation. Composition of two point symmetries of the given equation is the point symmetry for this equation. Therefore the set of point symmetries of the given equation forms the group, which is the important characteristic of the geometrical structure connected with this equation.

Let's consider one-parametric subgroups in the group of point symmetries of the equation (1.1). In order to do this we equip the map f with additional numeric parameter $t\in\mathbb{R}$. The property being a group make some limitations for the dependence of f_t on t:

$$f_0=\mathrm{id},\qquad\qquad f_{t+\tau}=f_t\circ f_\tau.\tag{2.9}$$

Let's write the transformations (2.9) in coordinates expanding them into the Tailor series in t centered at the point $t=0$:

$$\begin{aligned}&x\mapsto f^1_t(x,y)=x+t\cdot V(x,y)+\dots\,,\\ &y\mapsto f^2_t(x,y)=y+t\cdot U(x,y)+\dots\,.\end{aligned}\tag{2.10}$$

The quantities $Z^1 = V$ and $Z^2 = U$ in the expansions (2.10) appear to be the components of some vector field $\mathbf{Z}$ on the plane. Knowing this field, we can recover the transformations f_t if they form one-parametric group (2.9). Derivatives y' and y'' by the map (2.10) are transformed as follows:

$$y' \mapsto y' + t\cdot W_1 + \dots\,, \qquad\qquad y'' \mapsto y'' + t\cdot W_2 + \dots\,. \tag{2.11}$$

The quantities W_1 and W_2 are known as the components of *the first extension* and *the second extension* of the vector field $\mathbf{Z}$ (see [25] and [26]). Their forms are determined by the following formulas:

$$W_1 = U_{1.0} + (U_{0.1} - V_{1.0})\,y' - V_{0.1}\,(y')^2, \tag{2.12}$$

$$\begin{aligned} W_2 = {} & (U_{0.1} - 2\,V_{1.0} - 3\,V_{0.1}\,y')\,y'' + U_{2.0} + \\ & +(2\,U_{1.1} - V_{2.0})\,y' + (U_{0.2} - 2\,V_{1.1})\,(y')^2 - V_{0.2}\,(y')^3. \end{aligned} \tag{2.13}$$

Let's substitute (2.11) into the equation (1.1) and take into account the relationships (2.12) and (2.13). Moreover, we take into account the following expansions

$$\begin{aligned} &P \mapsto P + (P_{1.0}\,V + P_{0.1}\,U)\,t + \dots\,, && Q \mapsto Q + (Q_{1.0}\,V + Q_{0.1}\,U)\,t + \dots\,,\\ &R \mapsto R + (R_{1.0}\,V + R_{0.1}\,U)\,t + \dots\,, && S \mapsto S + (S_{1.0}\,V + S_{0.1}\,U)\,t + \dots\,, \end{aligned}$$

which are the consequences of (2.10). Then the condition of the existence of one-parametric group of point symmetries for the equation (1.1) is expressed in form of the system of four equations for two components of the vector field $\mathbf{Z}$:

$$\begin{aligned} U_{2.0} &= 3\,Q\,U_{1.0} + P_{0.1}\,U - P\,U_{0.1} + 2\,P\,V_{1.0} + P_{1.0}\,V,\\ -V_{0.2} &= 3\,R\,V_{0.1} + S_{1.0}\,V - S\,V_{1.0} + 2\,S\,U_{0.1} + S_{0.1}\,U,\\ 2\,U_{1.1} &= 3\,Q\,V_{1.0} + 3\,Q_{1.0}\,V + 3\,Q_{0.1}\,U + 6\,R\,U_{1.0} + 3\,P\,V_{0.1} + V_{2.0},\\ -2\,V_{1.1} &= 3\,R\,U_{0.1} + 3\,R_{0.1}\,U + 3\,R_{1.0}\,V + 6\,Q\,V_{0.1} + 3\,S\,U_{1.0} - U_{0.2}. \end{aligned} \tag{2.14}$$

This system of four equations (2.14) is overdetermined. It has a lot of differential consequences. First four of them can be solved with respect to the higher order derivatives $U_{3.0}$, $U_{2.1}$, $U_{1.2}$, and $U_{0.3}$. They are the following equations

$$\begin{aligned} 2\,U_{3.0} = {} & (2\,P_{0.1} + 18\,Q^2 + 6\,Q_{1.0} - 6\,P\,R)\,U_{1.0} + \\ & + (2\,P_{2.0} + 6\,Q\,P_{1.0} - 3\,P\,Q_{1.0})\,V + 3\,P\,V_{2.0} + \\ & + (2\,P_{1.1} - 3\,P\,Q_{0.1} + 6\,Q\,P_{0.1})\,U - 3\,P^2\,V_{0.1} + \\ & + (-2\,P_{1.0} - 6\,Q\,P_{0.0})\,U_{0.1} + (9\,Q\,P + 6\,P_{1.0})\,V_{1.0}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} 2\,U_{2.1} = {} & (2\,P_{1.1} + 9\,Q\,Q_{1.0} - 6\,P\,R_{1.0})\,V + \\ & + (-6\,P\,R_{0.1} + 9\,Q\,Q_{0.1} + 2\,P_{0.2})\,U + 3\,Q\,V_{2.0} + \\ & + (-6\,P\,S + 6\,Q_{0.1} + 18\,R\,Q)\,U_{1.0} - 6\,R\,P\,U_{0.1} + \\ & + (9\,Q^2 + 4\,P_{0.1})\,V_{1.0} + (-3\,Q\,P + 2\,P_{1.0})\,V_{0.1}, \end{aligned} \tag{2.16}$$

$$
\begin{aligned}
2\,U_{1.2} = {} & (9\,R\,Q_{1.0} + 4\,Q_{1.1} - 2\,R_{2.0} - 2\,S\,P_{1.0} - 4\,P\,S_{1.0})\,V + \\
& + (4\,Q_{0.2} - 2\,R_{1.1} - 2\,S\,P_{0.1} + 9\,R\,Q_{0.1} - 4\,P\,S_{0.1})\,U + 3\,R\,V_{2.0} + \\
& + (6\,R_{0.1} - 6\,S\,Q - 2\,S_{1.0} + 18\,R^2)\,U_{1.0} + (4\,P_{0.1} - 3\,P\,R)\,V_{0.1} + \\
& + (4\,Q_{0.1} - 6\,P\,S - 2\,R_{1.0})\,U_{0.1} + (9\,R\,Q + 4\,Q_{0.1} - 2\,R_{1.0})\,V_{1.0},
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
2\,U_{0.3} = {} & (18\,R\,R_{1.0} - 3\,S\,Q_{1.0} + 6\,R_{1.1} - 4\,S_{2.0} - 12\,Q\,S_{1.0})\,V + \\
& + (6\,R_{0.2} - 3\,S\,Q_{0.1} - 12\,Q\,S_{0.1} - 4\,S_{1.1} + 18\,R\,R_{0.1})\,U + \\
& + (2\,S_{0.1} + 12\,R\,S)\,U_{1.0} + (12\,Q_{0.1} - 6\,R_{1.0} - 3\,P\,S_{0.0})\,V_{0.1} + \\
& + (12\,R_{0.1} + 18\,R^2 - 8\,S_{1.0} - 24\,S\,Q)\,U_{0.1} + 9\,Q\,S\,V_{1.0} + 3\,S\,V_{2.0}.
\end{aligned}
\tag{2.18}
$$

In addition to (2.15), (2.16), (2.17), and (2.18), the equations (2.14) has next four differential consequences with derivatives $V_{3.0}$, $V_{2.1}$, $V_{1.2}$, and $V_{0.3}$:

$$
\begin{aligned}
2\,V_{0.3} = {} & (18\,R^2 - 6\,S\,Q - 6\,R_{0.1} - 2\,S_{1.0})\,V_{0.1} + \\
& + (6\,R\,S_{0.1} - 3\,S\,R_{0.1} - 2\,S_{0.2})\,U - 3\,S\,U_{0.2} + \\
& + (6\,R\,S_{1.0} - 2\,S_{1.1} - 3\,S\,R_{1.0})\,V - 3\,S^2\,U_{1.0} + \\
& + (9\,R\,S - 6\,S_{0.1})\,U_{0.1} + (-6\,R\,S + 2\,S_{0.1})\,V_{1.0},
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
2\,V_{1.2} = {} & (9\,R\,R_{0.1} - 6\,S\,Q_{0.1} - 2\,S_{1.1})\,U + \\
& + (9\,R\,R_{1.0} - 6\,S\,Q_{1.0} - 2\,S_{2.0})\,V - 3\,R\,U_{0.2} + \\
& + (18\,R\,Q - 6\,P\,S - 6\,R_{1.0})\,V_{0.1} - 6\,Q\,S\,V_{1.0} - \\
& - (3\,R\,S + 2\,S_{0.1})\,U_{1.0} + (9\,R^2 - 4\,S_{1.0})\,U_{0.1},
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
2\,V_{2.1} = {} & (2\,Q_{0.2} - 4\,S\,P_{0.1} - 4\,R_{1.1} - 2\,P\,S_{0.1} + 9\,Q\,R_{0.1})\,U + \\
& + (9\,Q\,R_{1.0} - 2\,P\,S_{1.0} + 2\,Q_{1.1} - 4\,R_{2.0} - 4\,S\,P_{1.0})\,V - 3\,Q\,U_{0.2} + \\
& + (2\,Q_{0.1} + 9\,R\,Q - 4\,R_{1.0})\,U_{0.1} + (2\,Q_{0.1} - 4\,R_{1.0} - 6\,P\,S)\,V_{1.0} + \\
& + (18\,Q^2 - 6\,Q_{1.0} - 6\,P\,R + 2\,P_{0,1})\,V_{0.1} - (4\,S_{1.0} + 3\,S\,Q)\,U_{1.0},
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
2\,V_{3.0} = {} & (4\,P_{0.2} - 3\,P\,R_{0.1} - 12\,R\,P_{0.1} - 6\,Q_{1.1} + 18\,Q\,Q_{0.1})\,U + \\
& + (18\,Q\,Q_{1.0} + 4\,P_{1.1} - 12\,R\,P_{1.0} - 6\,Q_{2.0} - 3\,P\,R_{1.0})\,V + \\
& + (6\,Q_{0.1} - 12\,R_{1.0} - 3\,P\,S)\,U_{1.0} + (12\,Q\,P - 2\,P_{1.0})\,V_{0.1} + \\
& + (18\,Q^2 - 12\,Q_{1.0} - 24\,P\,R + 8\,P_{0.1})\,V_{1.0} + 9\,R\,P\,U_{0.1} - 3\,P\,U_{0.2}.
\end{aligned}
\tag{2.22}
$$

Right hand sides of the equations (2.14), (2.15)–(2.18) and the last four equations (2.19), (2.20), (2.21), (2.22) contain the following functions: V, $V_{1.0}$, $V_{0.1}$, $V_{2.0}$, U, $U_{0.1}$, $U_{1.0}$, $U_{0.2}$. Lets form the vector-column Ψ with them. Then, using part of the above equations, we can obtain the complete system of Pfaff equations with respect to the components of the vector-column Ψ:

$$
\partial_x \Psi = \Lambda_x\,\Psi, \qquad\qquad \partial_y \Psi = \Lambda_y\,\Psi.
\tag{2.23}
$$

Here Λ_x and Λ_y are two square matrices 8×8. The compatibility condition for the system of linear equations (2.23) is expressed as a condition of commutating for two differential operators with matrix coefficients:

$$[\partial_x - \Lambda_x,\, \partial_y - \Lambda_y] = 0, \tag{2.24}$$

In the theory of integrability the equation (2.24) is known as *the equation of zero curvature*. Writing down it in components we find that it is reduced to

$$A = 0, \qquad\qquad B = 0, \tag{2.25}$$

where by A and B we denote the following quantities:

$$\begin{aligned} A &= P_{0.2} - 2\,Q_{1.1} + R_{2.0} + 2\,P\,S_{1.0} + S\,P_{1.0} -\\ &\qquad - 3\,P\,R_{0.1} - 3\,R\,P_{0.1} - 3\,Q\,R_{1.0} + 6\,Q\,Q_{0.1},\\ B &= S_{2.0} - 2\,R_{1.1} + Q_{0.2} - 2\,S\,P_{0.1} - P\,S_{0.1} +\\ &\qquad + 3\,S\,Q_{1.0} + 3\,Q\,S_{1.0} + 3\,R\,Q_{0.1} - 6\,R\,R_{1.0}. \end{aligned} \tag{2.26}$$

The quantities $\alpha_1 = A$ and $\alpha_2 = B$ appear to be the components of pseudocovectorial field of the weight 1. The quantities

$$\alpha^i = \sum_{k=1}^{2} d^{ik}\,\alpha_k \tag{2.27}$$

obtained from $\alpha_1 = A$ and $\alpha_2 = B$ by raising the lower index are the components of pseudovectorial field of the weight 2. The definition of pseudotensorial field just below can be found in [27].

Definition 2.2. Pseudotensorial field of the type (r, s) and weight m is an array of quantities $F^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}$ which under the change of variables (1.2) transforms as

$$F^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = (\det T)^m \sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1}\ldots\, S^{i_r}_{p_r}\; T^{q_1}_{j_1}\ldots\, T^{q_s}_{j_s}\; \tilde F^{p_1\ldots\, p_r}_{q_1\ldots\, q_s}. \tag{2.28}$$

Check of the rule (2.28) for the quantities α_k and for the quantities α^i in (2.27) can be done by direct calculations based on (2.7). The following lemma is a consequence of pseudotensorial character of the transformation rule for A and B.

Lemma 2.1. *Simultaneous vanishing of the parameters $A = 0$ and $B = 0$ for the initial equation* (1.1) *is equivalent to their vanishing $\tilde A = 0$ and $\tilde B = 0$ for the transformed equation* (1.3)*.*

Vanishing conditions (2.25) determine the case of maximal degeneration in the theory of the equations (1.1). This case is considered in the next section.

3. Case of maximal degeneration.

Let's take the conditions (2.25) to be fulfilled. These conditions are invariant with respect to the point transformations (see lemma 2.1 above). As a consequence of (2.25) we get the compatibility of the Pfaff equations (2.23). It is known that the set of solutions of complete compatible system of Pfaff equations is a finite dimensional linear vector-space over the real numbers $\mathbb{R}$. Its dimension coincides with the number of component of the vector-column Ψ, which is equal to eight.

The equations (2.14) are contained in the system of equations (2.23), therefore each solution of this system determines the vector field $\mathbf{Z}$ for some one-parametric group of point symmetries of the equation (1.1). It is known that the set of point symmetries of any equation is closed with respect to the mutual commutators of corresponding vector fields. Therefore the vector fields of point symmetries of the equation (1.1) form 8-dimensional Lie algebra over real numbers, provided the conditions (2.25) are fulfilled. Components of vector-column Ψ in some fixed point (e. g. the point $x = 0$ and $y = 0$) are the natural coordinates in this algebra. We shall not calculate the structural constants of this algebra now, but instead of this we shall prove the following theorem.

Theorem 3.1. *The equation* (1.1) *can be transformed to the form* $\tilde{y}'' = 0$ *by means of some point transformation if and only if the relationships* (2.25) *are fulfilled.*

Proof. Suppose that the equation (1.1) is reducible to the equation $\tilde{y}'' = 0$. The conditions $\tilde{P} = 0$, $\tilde{Q} = 0$, $\tilde{R} = 0$, and $\tilde{S} = 0$ for the coefficients of the transformed equation are written as the four differential equations for two functions $\tilde{x}(x, y)$ and $\tilde{y}(x, y)$ determining the appropriate point transformation (1.2) They are derived from (2.7). The equation $\tilde{P} = 0$ has the form:

$$(\tilde{x}_{2.0} + P\,\tilde{x}_{0.1})\,\tilde{y}_{1.0} = (\tilde{y}_{2.0} - P\,\tilde{y}_{0.1})\,\tilde{x}_{1.0}. \tag{3.1}$$

For the regular transformation (1.2) at least one of the derivatives $\tilde{x}_{1.0}$ or $\tilde{y}_{1.0}$ in (3.1) is nonzero. Therefore, after introducing an additional function $M(x, y)$, we can replace (3.1) by two equations:

$$\tilde{x}_{2.0} = M\,\tilde{x}_{1.0} - P\,\tilde{x}_{0.1}, \qquad\qquad \tilde{y}_{2.0} = M\,\tilde{y}_{1.0} - P\,yx_{0.1}. \tag{3.2}$$

In the similar way, the equation $\tilde{S} = 0$ can be broken into two equations after introducing another additional function $N(x, y)$:

$$\tilde{x}_{0.2} = S\,\tilde{x}_{1.0} - N\,\tilde{x}_{0.1}, \qquad\qquad \tilde{y}_{0.2} = S\,\tilde{y}_{1.0} - N\,yx_{0.1}. \tag{3.3}$$

Then the rest two equations $\tilde{Q} = 0$ and $\tilde{R} = 0$ can be used to determine the partial derivatives $\tilde{x}_{1.1}$ and $\tilde{y}_{1.1}$. For these two partial derivatives we obtain:

$$
\begin{aligned}
&\tilde{x}_{1.1}=\frac{3\,R-N}{2}\,\tilde{x}_{1.0}-\frac{3\,Q-N}{2}\,\tilde{x}_{0.1},\\
&\tilde{y}_{1.1}=\frac{3\,R-N}{2}\,\tilde{y}_{1.0}-\frac{3\,Q-N}{2}\,\tilde{y}_{0.1}.
\end{aligned}
\tag{3.4}
$$

As a result, for the function $\tilde{x}(x,y)$ we get the system of three differential equations, which can be expressed as the complete system of Pfaff equations (2.23) with respect to the vector column Ψ composed by two derivatives $\tilde{x}_{1.0}$ and $\tilde{x}_{0.1}$. The equations for the function $\tilde{y}(x,y)$ have exactly the same form.

By introducing the additional functions M and N we managed to separate dependent variables $\tilde{x}$ and $\tilde{y}$ in the equations $\tilde{P}=0$, $\tilde{Q}=0$, $\tilde{R}=0$, $\tilde{S}=0$, replacing them by two identical systems of equations. Being absolutely identical these two systems of equations have common compatibility condition, which is written as the matrix equation (2.24). In the present case this matrix equation is equivalent to the system of four equations for the functions M and N:

$$
\begin{aligned}
&M_{1.0}=3\,Q_{1.0}-2\,P_{0.1}+\frac{M^2}{2}+P\,N+3\,P\,R-\frac{9\,Q^2}{2},\\
&N_{1.0}=2\,Q_{0.1}-R_{1.0}-\frac{3\,M\,R}{2}-2\,P\,S+\frac{M\,N}{2}-\frac{3\,Q\,N}{2}+\frac{9\,Q\,R}{2},
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&M_{0.1}=2\,R_{1.0}-Q_{0.1}+\frac{3\,Q\,N}{2}+2\,S\,P-\frac{N\,M}{2}+\frac{3\,R\,M}{2}-\frac{9\,R\,Q}{2},\\
&N_{0.1}=3\,R_{0.1}-2\,S_{1.0}-\frac{N^2}{2}-S\,M-3\,S\,Q+\frac{9\,R^2}{2}.
\end{aligned}
\tag{3.6}
$$

The equations (3.5) and (3.6) form one more complete system of Pfaff equations. However this is the system of nonlinear equations. Theory of nonlinear Pfaff equations is given in the Appendix A (see section 13 below). By direct calculations based on this theory we find that compatibility condition for the equations (3.5) and (3.6) is exactly the condition (2.25). Thus we proved that condition (2.25) is necessary for the equation (1.1) can be reduced to the form $\tilde{y}''=0$.

In order to prove sufficiency of this condition, we simply do the above steps in the reverse order. From (2.25) we obtain the compatibility of the equations (3.5) and (3.6). By solving them we find $M(x,y)$ and $N(x,y)$, then substitute them into the equations (3.2), (3.3), and (3.4). This make them compatible. By solving the following initial value problem for these equations

$$
\tilde{x}_{1.0}\,\Bigr|_{\substack{x=a\\ y=b}}=1, \qquad \tilde{x}_{0.1}\,\Bigr|_{\substack{x=a\\ y=b}}=0,
\tag{3.7}
$$

$$
\tilde{y}_{1.0}\,\Bigr|_{\substack{x=a\\ y=b}}=0, \qquad \tilde{y}_{0.1}\,\Bigr|_{\substack{x=a\\ y=b}}=1.
\tag{3.8}
$$

we get the pair of functions $\tilde{x}(x, y)$ and $\tilde{y}(x, y)$. Due to initial data (3.7) and (3.8), we have the regularity of the point transformations (1.2) given by these functions in some neighborhood of the point $x = a$ and $y = b$. And finally, due to the equations (3.5) and (3.6), the transformed equation (1.3) has the form $\tilde{y}'' = 0$. $\square$

The theorem 3.1 just proved reduces all equations (1.1) satisfying the condition (2.25) to the trivial equation $y'' = 0$. Therefore we are to describe the algebra of point symmetries only for this trivial equation. Each element of this algebra can be represented in three forms: in form of one-parametric group of point transformations, in form of corresponding vector field with components (V, U), and in form of 8-dimensional constant vector

$$\Psi = (V, V_{1.0}, V_{0.1}, V_{2.0}, U, U_{0.1}, U_{1.0}, U_{0.2})$$

composed of the values of the above functions at some fixed point, e. g. at the point $x = 0$ and $y = 0$. First we describe the base elements of the algebra of point symmetries for the equation $y'' = 0$.

1. *Shift of argument*: $x \mapsto x + t$, $y \mapsto y$. Components of corresponding vector field are: $V = 1$, $U = 0$; and $\Psi = (1, 0, 0, 0, 0, 0, 0, 0)$.

2. *Shift of function value*: $x \mapsto x$, $y \mapsto y + t$. Components of corresponding vector field are: $V = 0$, $U = 1$; and $\Psi = (0, 0, 0, 0, 1, 0, 0, 0)$.

3. *Blowing up the argument*: $x \mapsto x\,e^t$, $y \mapsto y$. Components of corresponding vector field are: $V = x$, $U = 0$; and $\Psi = (0, 1, 0, 0, 0, 0, 0, 0)$.

4. *Blowing up the function value*: $x \mapsto x$, $y \mapsto y\,e^t$. Components of corresponding vector field are: $V = 0$, $U = y$; and $\Psi = (0, 0, 0, 0, 0, 1, 0, 0)$.

5. *Shift of argument with zoom*: $x \mapsto x + y\,t$, $y \mapsto y$. Components of corresponding vector field are: $V = y$, $U = 0$; and $\Psi = (0, 0, 1, 0, 0, 0, 0, 0)$.

6. *Shift of function value with zoom*: $x \mapsto x$, $y \mapsto y + x\,t$. Components of corresponding vector field are: $V = 0$, $U = x$; and $\Psi = (0, 0, 0, 0, 0, 0, 1, 0)$.

7. *Inversion of the argument*. Components of corresponding vector field are: $V = x^2/2$, $U = x\,y/2$; and $\Psi = (0, 0, 0, 1, 0, 0, 0, 0)$. One-parametric group of point transformations is given by

$$x \mapsto \frac{2\,x}{2 - x\,t}, \qquad\qquad y \mapsto \frac{2\,y}{2 - x\,t}.$$

8. *Inversion of the function value*. Components of corresponding vector field are: $V = x\,y/2$, $U = y^2/2$; and $\Psi = (0, 0, 0, 0, 0, 0, 0, 1)$. One-parametric group of point transformations is given by

$$x \mapsto \frac{2\,x}{2 - y\,t}, \qquad\qquad y \mapsto \frac{2\,y}{2 - y\,t}.$$

Theorem 3.2. *Algebra of point symmetries of the equation* (1.1)*, satisfying the condition* (2.25)*, is isomorphic to the special linear Lie algebra* $\mathrm{sl}(3, \mathbb{R})$*.*

Proof. Because of the theorem 3.1 we should prove this statement only for the equation $y''=0$. In order to do this, let's consider the map

$$\left\|\begin{matrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\\ \psi_6\\ \psi_7\\ \psi_8\end{matrix}\right\| \mapsto \left\|\begin{matrix}\dfrac{2\,\psi_2-\psi_6}{3} & \psi_3 & \psi_1\\[2ex] \psi_7 & \dfrac{2\,\psi_6-\psi_2}{3} & \psi_5\\[2ex] -\dfrac{\psi_4}{2} & -\dfrac{\psi_8}{2} & -\dfrac{\psi_2+\psi_6}{3}\end{matrix}\right\|. \tag{3.9}$$

Now by means of direct calculations one can show that (3.9) is the very map that implements the isomorphism of the algebra of point symmetries of the equation $y''=0$ and the matrix algebra $\mathrm{sl}(3,\mathbb{R})$. $\square$

In terms of matrices from the group $\mathrm{SL}(3,\mathbb{R})$ we can describe all point transformations (1.2) that transform the equation $y''=0$ into the same equation $\tilde{y}''=0$:

$$\tilde{x}=\frac{g^1_1\,x+g^1_2\,y+g^1_3}{g^3_1\,x+g^3_2\,y+g^3_3}, \qquad\qquad \tilde{y}=\frac{g^2_1\,x+g^2_2\,y+g^2_3}{g^3_1\,x+g^3_2\,y+g^3_3}. \tag{3.10}$$

Formulas (3.10) can be derived by solving the equations (3.5) and (3.6) for the case $P=Q=R=S=0$. Then the obtained functions M and N should be substituted into the equations (3.2)–(3.4). Solutions of the latter ones are the functions $\tilde{x}(x,y)$ and $\tilde{y}(x,y)$ of the form (3.10).

4. Case of general position.

Now suppose that parameters A and B in (2.26) do not vanish simultaneously. Therefore we have two extra compatibility equations for the components of the vector field of the point symmetry

$$\begin{aligned} &2\,V_{1.0}\,A+V\,A_{1.0}+U_{1.0}\,B+U_{0.1}\,A+U\,A_{0.1}=0,\\ &V\,B_{1.0}+2\,U_{0.1}\,B+U\,B_{0.1}+V_{0.1}\,A+V_{1.0}\,B=0.\end{aligned} \tag{4.1}$$

They appear as the following compatibility conditions for the third order derivatives from (2.15), (2.16), (2.20), and (2.22):

$$\frac{\partial U_{3.0}}{\partial y}=\frac{\partial U_{2.1}}{\partial x}, \qquad\qquad \frac{\partial V_{0.3}}{\partial x}=\frac{\partial V_{1.2}}{\partial y}.$$

Other equations of this sort add nothing new to (4.1).

The equations (4.1) are not unconditionally solvable respective to any one of derivatives $U_{1.0}$, $U_{0.1}$, $V_{1.0}$, and $V_{0.1}$ in them. But we can overcome this difficulty. Let's remember that the quantities $\alpha^1=B$ and $\alpha^2=-A$ are the components of the pseudovectorial field of the weight 2. This means that we can find local coordinates on the plane such that $A=0$ and $B\neq 0$. In order to do this, we should choose some vector field collinear to α and then straighten it by changing for the appropriate local coordinates.

Now let's do one more point transformation that preserve the condition $A = 0$. It should be of the following form

$$\tilde{x} = \tilde{x}(x, y), \qquad \tilde{y} = y. \tag{4.2}$$

The transformation rule for the pseudovector α under the point transformation (4.2) can be written in matrix form:

$$\left\| \begin{matrix} \tilde{B} \\ 0 \end{matrix} \right\| = (\det T)^2 \left\| \begin{matrix} x_{1.0} & x_{0.1} \\ 0 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} B \\ 0 \end{matrix} \right\|, \tag{4.3}$$

(see definition 2.2 and the formulas (2.1) and (2.28)). From (4.3) we obtain $\tilde{B} = \tilde{x}_{1.0}\, B$. By means of proper choice of the function $\tilde{x}(x, y)$ in (4.2) we can get $\tilde{B} = 1$. Therefore we may take the following condition

$$\alpha^2 = A = 0, \qquad \alpha^1 = B = 1 \tag{4.4}$$

to be true from the very beginning. Now substituting (4.4) into (4.1) we can solve these equations with respect to the following derivatives of the first order:

$$U_{1.0} = 0, \qquad V_{1.0} = -2\, U_{0.1}. \tag{4.5}$$

From (4.5) we derive new compatibility conditions

$$U_{2.0} = \frac{\partial U_{1.0}}{\partial x}, \qquad U_{1.1} = \frac{\partial U_{1.0}}{\partial y}. \tag{4.6}$$

Taking into account the equations (2.14), from the first equation (4.6) we get

$$P\, U_{0.1} = \frac{1}{5}\, P_{1.0}\, V + \frac{1}{5}\, P_{0.1}\, U. \tag{4.7}$$

The second equation (4.6) can be solved with respect to the derivative $V_{2.0}$:

$$V_{2.0} = 6\, Q\, U_{0.1} - 3\, Q_{1.0}\, V - 3\, Q_{0.1}\, U - 3\, P\, V_{0.1}. \tag{4.8}$$

Now, taking into account (4.8) and (2.14), from (4.5) we can derive next two relationships, which are the compatibility conditions like (4.6):

$$V_{2.0} = \frac{\partial V_{1.0}}{\partial x}, \qquad V_{1.1} = \frac{\partial V_{1.0}}{\partial y}. \tag{4.9}$$

In explicit form, first of the compatibility equations (4.9) is written as follows:

$$P\, V_{0.1} = 2\, Q\, U_{0.1} - Q_{1.0}\, V - Q_{0.1}\, U, \tag{4.10}$$

Second compatibility equation (4.9) can be solved with respect to the derivative of the second order $U_{0.2}$:

$$U_{0.2} = \frac{6}{5}\, Q\, V_{0.1} + \frac{3}{5}\, R\, U_{0.1} + \frac{3}{5}\, R_{1.0}\, V + \frac{3}{5}\, R_{0.1}\, U. \tag{4.11}$$

Further study of determining equations for the vector field of point symmetry for the equation (1.1) is subdivided into several cases. *The case of general position* is distinguished by the condition $P \neq 0$.

Local coordinates satisfying the conditions (4.4) are not determined uniquely: change from some particular coordinates with this property to another ones is given by the following relationships

$$\tilde{x} = \frac{x}{h'(y)^2} + g(y), \qquad\qquad \tilde{y} = h(y), \tag{4.12}$$

where $g(y)$ and $h(y)$ are two arbitrary functions of one variable, and where $h'(y) \neq 0$. Now let's write down the transition matrices for the change of variables (4.12):

$$T = \begin{Vmatrix} \dfrac{1}{h'^2} & -\dfrac{2\,x\,h''}{h'^3} + g' \\ \\ 0 & h' \end{Vmatrix}. \qquad\qquad S = \begin{Vmatrix} h'^2 & \dfrac{2\,x\,h''}{h'^2} - g'\,h' \\ \\ 0 & \dfrac{1}{h'} \end{Vmatrix}. \tag{4.13}$$

Transformation rules for the coefficients of the equation (1.1) are defined by (2.7). However, due to the special form of point transformation (4.12), and due to the special form of transition matrices (4.13), these transformation rules are substantially simplified. The most simple is the rule for coefficient P:

$$P = \frac{\tilde{P}}{h'^5} = (\det T)^5\,\tilde{P}. \tag{4.14}$$

The transformation rule (4.14) let us define the pseudoscalar field F of the unit weight 1. In order to do it we set

$$F^5 = -P. \tag{4.15}$$

in some special local coordinates, where the conditions (4.4) are fulfilled.

Apart from the pseudoscalar field F from (4.15), we introduce four quantities β_1, β_2, β^1, and β^2. Let's define them by formulas

$$\beta_1 = 3\,P, \qquad\qquad \beta_2 = 3\,Q, \tag{4.16}$$

$$\beta^1 = 3\,Q, \qquad\qquad \beta^2 = -3\,P \tag{4.17}$$

in the above special coordinates. One can easily check that by the transformations (4.12) the quantities (4.16) and (4.17) are transformed as pseudocovectorial field of the weight 3 and as pseudovectorial field of the weight 4 respectively.

Using the pseudoscalar field F, we define the quantities φ_i as logarithmical derivatives of F:

$$\varphi_i = -\frac{\partial \ln F}{\partial x^i}. \tag{4.18}$$

By the point transformations (4.12) these quantities are transformed according to the following rule:

$$\varphi_i = \sum_{j=1}^{2} T^j_i\,\tilde{\varphi}_j - \sigma_i. \tag{4.19}$$

Due to the relationships (4.19), we can use the quantities φ_i in order to construct the components of affine connection on the base of quantities (2.5):

$$\varGamma^k_{ij} = \theta^k_{ij} - \frac{\varphi_i\,\delta^k_j + \varphi_j\,\delta^k_i}{3} \tag{4.20}$$

In the case of general position $F \neq 0$. Therefore we can use this field to construct the pair of vector fields $\mathbf{X}$ and $\mathbf{Y}$ with the components:

$$X^i = \frac{\alpha^i}{F^2}, \qquad\qquad Y^i = \frac{\beta^i}{F^4}. \tag{4.21}$$

Vectorial fields (4.21) form the moving frame on the plane. Let's calculate the components of connection (4.20) related to this frame. They are defined as the coefficients in the following expansions:

$$\begin{aligned} &\nabla_{\mathbf{X}}\mathbf{X} = \Gamma^1_{11}\,\mathbf{X} + \Gamma^2_{11}\,\mathbf{Y}, &&\nabla_{\mathbf{X}}\mathbf{Y} = \Gamma^1_{12}\,\mathbf{X} + \Gamma^2_{12}\,\mathbf{Y},\\ &\nabla_{\mathbf{Y}}\mathbf{X} = \Gamma^1_{21}\,\mathbf{X} + \Gamma^2_{21}\,\mathbf{Y}, &&\nabla_{\mathbf{Y}}\mathbf{Y} = \Gamma^1_{22}\,\mathbf{X} + \Gamma^2_{22}\,\mathbf{Y}. \end{aligned} \tag{4.22}$$

In contrast to the quantities $\varGamma^k_{ij}$ in (4.20), the coefficients Γ^k_{ij} from (4.22) do not change by the point transformations (4.12). These are scalar fields, *i. e. scalar invariants* for the differential equation (1.1). Let's numerate them as follows: $I_1 = \Gamma^1_{11}$, $I_2 = \Gamma^2_{11}$, $I_3 = \Gamma^1_{12}$, $\ldots$, $I_8 = \Gamma^2_{22}$. The list of scalar invariants of the equation (1.1) can be continued. By differentiating these invariants along the vector fields $\mathbf{X}$ and $\mathbf{Y}$, we obtain some new invariants $I_9 = \mathbf{X}I_1$, $\ldots$, $I_{16} = \mathbf{X}I_8$, $I_{17} = \mathbf{Y}I_1$, $\ldots$, $I_{24} = \mathbf{Y}I_8$. Repeating this procedure, we shall get 16 new invariants in each step. The global structure of all these invariants I_1, I_2, I_3, $\ldots$ determines the branching into three different cases:

1. in the infinite series of invariants $I_k(x, y)$ one can find two functionally independent ones;
2. invariants $I_k(x, y)$ are functionally dependent, but not all of them are constants;
3. all invariants $I_k(x, y)$ are constants.

Let's write down the explicit expressions for the first eight invariants in the special coordinates, where the conditions (4.4) hold:

$$I_2 = \frac{1}{3}, \qquad\qquad I_6 = \frac{1}{3}\,\frac{F_{1.0}}{F^3}. \tag{4.23}$$

Formulas for I_3 and I_8 are a little more huge:

$$\begin{aligned} &I_3 = 3\,\frac{Q_{1.0}}{F^4} - 14\,\frac{Q\,F_{1.0}}{F^5} + 3\,\frac{Q^2}{F^4} + 3\,F\,R + F_{0.1},\\ &I_8 = 5\,\frac{Q\,F_{1.0}}{F^5} - 3\,\frac{Q^2}{F^4} + 5\,F_{0.1} - 3\,F\,R. \end{aligned} \tag{4.24}$$

Formula for the invariant I_7 is even more huge than previous ones:

$$\begin{aligned} I_7 = 9\,\frac{Q\,Q_{1,0}}{F^6} &- 45\,\frac{Q^2\,F_{1,0}}{F^7} + 18\,\frac{Q^3}{F^6} +\\ &+ 27\,\frac{Q\,R}{F} - 45\,\frac{Q\,F_{0,1}}{F^2} + 9\,\frac{Q_{0,1}}{F} + 9\,F^4\,S. \end{aligned} \tag{4.25}$$

For calculating other invariants I_1, I_4, and I_5 we needn't special formulas different from (4.23), (4.24), and (4.25). They are expressed through I_6 and I_8:

$$I_1 = -4\,I_6, \qquad I_4 = 4\,I_6, \qquad I_5 = -I_8. \tag{4.26}$$

Due to the relationships (4.26), the number of basic invariants in the series I_1, I_2, I_3, I_4, ... is equal to four. These are I_3, I_6, I_7 and I_8. Other invariants can be derived from the basic ones.

5. Point symmetries in the case of general position.

In the case of general position $P = -F^5 \neq 0$. Therefore the equations (4.7) and (4.10) can be solved with respect to the derivatives U_{01} and $V_{0.1}$. This gives

$$\begin{aligned} &U_{0.1} = \frac{F_{1.0}}{F}\,V + \frac{F_{0.1}}{F}\,U,\\ &V_{0.1} = \frac{Q_{1.0}\,F - 2\,F_{1.0}\,Q}{F^6}\,V + \frac{Q_{0.1}\,F - 2\,F_{0.1}\,Q}{F^6}\,U. \end{aligned} \tag{5.1}$$

The equations (5.1) are complemented by the equations (4.5), which are written in the the following form now:

$$U_{1.0} = 0, \qquad V_{1.0} = -\frac{2\,F_{1.0}}{F}\,V - \frac{2\,F_{0.1}}{F}\,U. \tag{5.2}$$

From (5.1) and (5.2) we derive a series of new compatibility conditions:

$$U_{1.1} = \frac{\partial U_{0.1}}{\partial x}, \qquad U_{0.2} = \frac{\partial U_{0.1}}{\partial y}, \tag{5.3}$$

$$V_{1.1} = \frac{\partial V_{0.1}}{\partial x}, \qquad V_{0.2} = \frac{\partial V_{0.1}}{\partial y}. \tag{5.4}$$

The four relationships (5.3) and (5.4) can be written in terms of the scalar invariants I_3, I_6, I_7 and I_8 considered above:

$$\begin{aligned} &\frac{\partial I_3}{\partial x}\,V + \frac{\partial I_3}{\partial y}\,U = 0, &&\frac{\partial I_6}{\partial x}\,V + \frac{\partial I_6}{\partial y}\,U = 0,\\ &\frac{\partial I_7}{\partial x}\,V + \frac{\partial I_7}{\partial y}\,U = 0, &&\frac{\partial I_8}{\partial x}\,V + \frac{\partial I_8}{\partial y}\,U = 0. \end{aligned} \tag{5.5}$$

Next four second order compatibility conditions, which are not written in (5.3) and (5.4), add nothing new to the equations (5.5).

One more immediate consequence of the equations (5.1) and (5.2) is the permutability of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ (defined by (4.21)) with the vector field $\mathbf{Z}$ of the point symmetry for the equation (1.1):

$$[\mathbf{Z}, \mathbf{X}] = 0, \qquad [\mathbf{Z}, \mathbf{Y}] = 0. \tag{5.6}$$

The equations (5.5) have transparent geometrical interpretation. They mean that basic invariants I_3, I_6, I_7, and I_8 are constants along the integral lines of the vector field of point symmetry $\mathbf{Z}$. From (4.26) it follows that the same assertion is true for I_1, I_4, and I_5. For the invariant I_2 this is trivial, since I_2 is a constant. As a result, we conclude:

$$\mathbf{Z}I_k = 0 \quad \text{for} \quad k = 1, \ldots, 8. \tag{5.7}$$

Using the permutability equations (5.6), we can expand the relationships (5.7) for the whole infinite series of invariants I_k. Let's do this by induction. Remember that $I_{k+8} = \mathbf{X}I_k$ and $I_{k+16} = \mathbf{Y}I_k$. Then

$$\mathbf{Z}I_{k+8} = \mathbf{Z}\mathbf{X}I_k = [\mathbf{Z},\mathbf{X}]I_k + \mathbf{X}\mathbf{Z}I_k = 0,$$

$$\mathbf{Z}I_{k+16} = \mathbf{Z}\mathbf{Y}I_k = [\mathbf{Z},\mathbf{Y}]I_k + \mathbf{Y}\mathbf{Z}I_k = 0.$$

As an immediate consequence of the relationships $\mathbf{Z}I_k = 0$, for the whole series of invariants we get the following theorem.

Theorem 5.1. *In the case of general position $F \neq 0$ the algebra of point symmetries for the equation* (1.1) *with functionally independent invariants is trivial.*

Indeed, the relationships $\mathbf{Z}I_p = 0$ and $\mathbf{Z}I_q = 0$ for two functionally independent invariants I_p and I_q lead to the vanishing of the vector field $\mathbf{Z} = 0$.

Theorem 5.2. *In the case of general position $F \neq 0$, when all invariants I_k are functionally dependent, but not all of them are constants, the algebra of point symmetries of the equation* (1.1) *is unidimensional.*

Proof. In the case of functionally dependent invariants all equations $\mathbf{Z}I_k = 0$ are the consequences of one of them. Let's consider only the equations (5.1) and (5.2), forgetting for a while all other equations. They form complete system of linear Pfaff equations. Let's impose an extra restriction $\mathbf{Z}I_k = 0$ on them, where I_k is some nonconstant invariant. Thus we obtain the complete system of linear Pfaff equations with restriction. Theory of such equations is given in the Appendix B (see section 14 below). According to this theory, compatibility condition for the equations (5.1) and (5.2) is formed by the following equations

$$\mathbf{Z}I_6 = 0, \qquad\qquad \mathbf{Z}I_3 + \mathbf{Z}I_8 = 0, \tag{5.8}$$

which hold due to the restriction $\mathbf{Z}I_k = 0$. Therefore the equations (5.1) and (5.2) with restriction $\mathbf{Z}I_k = 0$ has unidimensional space of solutions.

Second order equations (5.4), (5.3), (4.11), (4.8), and (2.14) are compatible with (5.1) and (5.2). Therefore the dimension of the algebra of point symmetries of the equation (1.1) in this case is equal to one. $\square$

In case of identically constant invariants the equations (5.5) are (5.8) identically zero and we have no restrictions for the Pfaff equations (5.1) and (5.2). They are compatible and the space of their solutions is two-dimensional.

Theorem 5.3. *In the case of general position $F \neq 0$, when all invariants I_k are constant, the algebra of point symmetries of the equation* (1.1) *is two-dimensional.*

For the arbitrary values of invariants this two-dimensional Lie algebra is integrable but not commutative. However, if

$$I_6 = 0, \qquad\qquad I_3 + I_8 = 0, \tag{5.9}$$

then this algebra is Abelian. The relationships (5.9) are derived by direct computation of structural constants.

6. First case of intermediate degeneration.

In previous section we chose special coordinates, where the conditions (4.4) are fulfilled, and we considered the case of general position, when $P \neq 0$. Now let's suppose that $P = 0$. In this case, equation (4.7) is identically zero, and the equation (4.10) is rewritten as follows:

$$2\,Q\,U_{0.1} = Q_{1.0}\,V + Q_{0.1}\,U. \tag{6.1}$$

The solvability of the equation (6.1) with respect to the derivative $U_{0.1}$ is determined by the condition $Q \neq 0$. For the case $P = 0$, the quantities (4.19) and fields (4.21) are not defined. This require another theory of invariants especially for this case.

Lets consider the point transformations (4.12). From the relationship (4.14) we see that the condition $P = 0$ is preserved by these transformations. For $P = \tilde{P} = 0$, the quantity Q is transformed as follows:

$$Q = \frac{\tilde{Q}}{h'^2} = (\det T)^2\,\tilde{Q}. \tag{6.2}$$

Due to (6.2), the condition $Q \neq 0$ is also preserved by the transformations (4.12). In this case the quantity Q behave as pseudoscalar field of the weight 2.

Let's solve the equation (6.1) with respect to the derivative $U_{0.1}$ and then consider the differential consequences of it:

$$U_{1.1} = \frac{\partial U_{0.1}}{\partial x}, \qquad\qquad U_{0.2} = \frac{\partial U_{0.1}}{\partial y}. \tag{6.3}$$

First of the relationships (6.3) does not contain the derivative $V_{0.1}$. It is written as:

$$\frac{1}{2}\,\frac{Q_{2.0}}{Q}\,V - \frac{Q_{1.0}^2}{Q^2}\,V + \frac{1}{2}\,\frac{Q_{1.1}}{Q}\,U - \frac{Q_{1.0}\,Q_{0.1}}{Q^2}\,U = 0. \tag{6.4}$$

The second relationship contains the derivative $V_{0.1}$. It looks like

$$\begin{aligned}&\left(\frac{Q_{1.0}}{Q} - \frac{12}{5}\,Q\right)V_{0.1} - \frac{1}{2}\,\frac{Q_{0.1}\,Q_{1.0}}{Q^2}\,V + \frac{Q_{1.1}}{Q}\,V - \frac{6}{5}\,R_{0.1}\,U -\\ &-\frac{1}{2}\,\frac{Q_{0.1}^2}{Q^2}\,U + \frac{Q_{0.2}}{Q}\,U - \frac{6}{5}\,R_{1.0}\,V - \frac{3}{5}\,\frac{R\,Q_{1.0}}{Q}\,V - \frac{3}{5}\,\frac{R\,Q_{0.1}}{Q}\,U = 0.\end{aligned} \tag{6.5}$$

The solvability of the equation (6.5) with respect to the derivative $V_{0.1}$ depends on the following quantity:

$$M = Q_{1.0} - \frac{12}{5}\,Q^2. \tag{6.6}$$

The condition $M \neq 0$ leads to $Q \neq 0$. This condition $M \neq 0$, combined with $P = 0$, determines *the first case of intermediate degeneration*.

By means of direct calculations one can check that for $P = 0$, the quantity M in (6.6), by point transformations (4.12), is transformed as the pseudoscalar field of the weight 4 :

$$M = \frac{\tilde{M}}{h'^4} = (\det T)^4\,\tilde{M}. \tag{6.7}$$

Due to (6.7), the condition $M \neq 0$ is preserved by the transformations (4.12). The quantity M in this section plays the same role as the quantity P in the case of general position. However, the weights of the fields Q and M are even. This prevent us from defining the field of the unit weight by taking the root of the appropriate power. This circumstance make more difficult the theory of invariants in this case, but one invariant can be written right now:

$$I_1 = \frac{M}{Q^2}. \tag{6.8}$$

In order to construct an affine connection, in previous section we used the quantities (4.18). Here in place of them we introduce the following ones:

$$\varphi_1 = -\frac{6}{5}\,Q, \qquad\qquad \varphi_2 = -\frac{3}{5}\,R. \tag{6.9}$$

It is easy to check that the quantities (6.9) obey the rule (4.19) under the point transformations (4.12). This lets us construct an affine connection by formula

$$\Gamma^k_{ij} = \theta^k_{ij} - \frac{\varphi_i\,\delta^k_j + \varphi_j\,\delta^k_i}{3}, \tag{6.10}$$

where the quantities θ^k_{ij} are defined according to (2.4). Despite the coincidence of formulas (6.10) and (4.20) they determine two different connections. This difference is due to the difference in parameters φ_i.

The quantities (6.9) and the components of affine connection let us define the operation of covariant differentiation for the pseudotensorial fields. In case of the field of type (r,s) and weight m we set

$$\begin{aligned}\nabla_k F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{\partial F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^k} + \sum^{r}_{n=1}\sum^{2}_{v_n=1}\Gamma^{i_n}_{k\,v_n}\,F^{i_1\ldots\,v_n\ldots\,i_r}_{j_1\ldots\,j_s} -\\ &-\sum^{s}_{n=1}\sum^{2}_{w_n=1}\Gamma^{w_n}_{k\,j_n}\,F^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s} + m\,\varphi_k\,F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}.\end{aligned} \tag{6.11}$$

For $m=0$, this formula (6.11) coincides with the standard formula for covariant derivatives (more details see in [27]). We apply this operation of covariant differentiation to the pseudovectorial field α of the weight 2 with the components (4.4) and to the pseudovectorial field γ with the following components

$$\gamma^1=-2\,R_{1.0}+3\,Q_{0.1}+\frac{6}{5}\,Q\,R,\qquad\qquad \gamma^2=M. \tag{6.12}$$

The condition of non-collinearity of the fields α and γ is exactly the condition $M\neq 0$, which is fulfilled for the present case. Therefore we can write the expansions analogous to (4.22):

$$\begin{aligned}&\nabla_{\alpha}\alpha=\Gamma^1_{11}\,\alpha+\Gamma^2_{11}\,\gamma, &&\nabla_{\alpha}\gamma=\Gamma^1_{12}\,\alpha+\Gamma^2_{12}\,\gamma,\\ &\nabla_{\gamma}\alpha=\Gamma^1_{21}\,\alpha+\Gamma^2_{21}\,\gamma, &&\nabla_{\gamma}\gamma=\Gamma^1_{22}\,\alpha+\Gamma^2_{22}\,\gamma.\end{aligned} \tag{6.13}$$

The field γ with components (6.12) is of weight 3. This determines the weights of the pseudoscalar fields Γ^k_{ij} in the expansions (6.13): the field Γ^2_{11} has the weight 1, fields Γ^1_{11}, Γ^2_{12}, and Γ^2_{21} are of weight 2, fields Γ^1_{12}, Γ^1_{21}, and Γ^2_{22} have the weight 3, and finally, the field Γ^1_{22} has the weight 4. Now we give explicit formulas for these fields. Part of them are very simple

$$\begin{aligned}&\Gamma^2_{11}=0, &&\Gamma^1_{11}=\Gamma^2_{21}=-\frac{3}{5}\,Q,\\ &\Gamma^1_{21}=0, &&\Gamma^2_{12}=\frac{M_{1.0}}{M}-\frac{21}{5}\,Q.\end{aligned} \tag{6.14}$$

Formulas for the pair of fields Γ^1_{12} and Γ^2_{22} differ from each other only by sign:

$$\begin{aligned}\Gamma^2_{22}=-\Gamma^1_{12}&=M_{0.1}-\frac{72}{5}\,Q\,Q_{0.1}+\frac{48}{5}\,Q\,R_{1.0}-\frac{12}{5}\,R\,M-\\ &-\frac{144}{25}\,R\,Q^2-2\,\frac{R_{1.0}\,M_{1.0}}{M}+3\,\frac{Q_{0.1}\,M_{1.0}}{M}+\frac{6}{5}\,\frac{Q\,R\,M_{1.0}}{M}.\end{aligned} \tag{6.15}$$

Formula for Γ^1_{22} appears to be the most huge. It has the form:

$$\begin{aligned}\Gamma^1_{22}&=\frac{24}{5}\,\frac{R_{1.0}\,Q\,R\,M_{1.0}}{M}-\frac{36}{5}\,\frac{Q_{0.1}\,Q\,R\,M_{1.0}}{M}-2\,M\,R_{1.1}+\\ &+3\,M\,Q_{0.2}+M^2\,S-4\,\frac{R^2_{1.0}\,M_{1.0}}{M}-9\,\frac{Q^2_{0.1}\,M_{1.0}}{M}-\\ &-\frac{324}{5}\,R_{1.0}\,Q\,Q_{0.1}-\frac{648}{25}\,R_{1.0}\,R\,Q^2+\frac{972}{25}\,Q_{0.1}\,R\,Q^2-\\ &-\frac{12}{5}\,Q\,R\,M_{0.1}-\frac{36}{25}\,\frac{Q^2\,R^2\,M_{1.0}}{M}+12\,\frac{R_{1.0}\,Q_{0.1}\,M_{1.0}}{M}+\\ &+\frac{126}{25}\,Q\,R^2\,M-\frac{42}{5}\,R_{1.0}\,R\,M+\frac{69}{5}\,Q_{0.1}\,R\,M+\\ &+\frac{6}{5}\,M\,Q\,R_{0.1}+4\,R_{1.0}\,M_{0.1}+\frac{108}{5}\,Q\,R^2_{1.0}-\\ &-6\,Q_{0.1}\,M_{0.1}+\frac{243}{5}\,Q\,Q^2_{0.1}+\frac{972}{125}\,Q^3\,R^2.\end{aligned} \tag{6.16}$$

The quantities φ_i themselves determine the skew-symmetric tensor field ω_{ij} of the weight zero with the following components:

$$\omega_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}. \tag{6.17}$$

Upon contracting (6.17) with the matrix d^{ij} from (2.6), we get the pseudoscalar field Ω of the following form

$$\Omega = \frac{5}{6} \sum_{i=1}^{2} \sum_{j=1}^{2} \omega_{ij}\, d^{ij} = R_{1.0} - 2\, Q_{0.1}. \tag{6.18}$$

Field (6.18) has the weight 1. This field is used in order to determine one more scalar invariant of the equation (1.1):

$$I_2 = \frac{\Omega^2}{Q} = \frac{R_{1.0}^2 - 4\, R_{1.0}\, Q_{0.1} + 4\, Q_{0.1}^2}{Q}. \tag{6.19}$$

Third invariant I_3 is determined by the field Γ^1_{22} from (6.16). It has the form:

$$I_3 = \frac{\Gamma^1_{22}\, Q^2}{M^2}. \tag{6.20}$$

Invariants (6.8), (6.19), and (6.20) are basic ones. Other invariants are derived from them by means of differentiation along the fields α and γ according to the rule:

$$I_{k+3} = \frac{\nabla_\alpha I_k}{Q}, \qquad\qquad I_{k+6} = \frac{\bigl(\nabla_\gamma I_k\bigr)^2}{Q^3}. \tag{6.21}$$

Applying the rule (6.21) repeatedly step by step, we get 6 new invariants in each step. Moreover, we have:

$$I_1\, \Gamma^2_{12} = I_4\, Q - \frac{3}{5}\, I_1\, Q - 2\, I_1^2\, Q, \tag{6.22}$$

$$\begin{aligned} &\bigl(I_1\, \Gamma^2_{22}\bigr)^4 + \bigl(I_7\, Q^3\bigr)^2 + \bigl(16\, I_2\, Q^3\, I_1{}^4\bigr)^2 = \\ &\qquad = 32\, I_7\, Q^6\, I_2\, I_1{}^4 + 2\,\bigl(I_7\, Q^3 + 16\, I_2\, Q^3\, I_1{}^4\bigr)\bigl(I_1\, \Gamma^2_{22}\bigr)^2. \end{aligned} \tag{6.23}$$

The relationships (6.22) and (6.23) bind the invariants I_4 and I_7 with the fields Γ^1_{12} and Γ^2_{22} from (6.14) and (6.15).

Now we return to the study of the point symmetries of the equation (1.1) taking into account the above theory of invariants. Using the notation (6.6) and keeping in mind that $M \neq 0$, we solve the equation (6.5) with respect to the derivative $V_{0.1}$. As a result, we complement the equation (6.4) with two differential consequences of the equation (6.5). They have the form (5.4). These two equations can be reduced to the following relationships for the invariants I_2 and I_3:

$$\frac{\partial I_2}{\partial x}\, V + \frac{\partial I_2}{\partial y}\, U = 0, \qquad\qquad \frac{\partial I_3}{\partial x}\, V + \frac{\partial I_3}{\partial y}\, U = 0. \tag{6.24}$$

The equation (6.4) itself can be transformed to the analogous relationship for I_1:

$$\frac{\partial I_1}{\partial x}\,V+\frac{\partial I_1}{\partial y}\,U=0. \tag{6.25}$$

From (6.24) and (6.25) one can derive similar relationships for all invariants in the series I_1, I_2, I_3, $\ldots$ constructed according to the rule (6.21). Proof of this fact is based on the relationship (5.6) for the following two vector fields:

$$\mathbf{X}=\frac{\alpha}{Q}, \qquad\qquad \mathbf{Y}=\frac{\gamma}{Q^{3/2}}.$$

The relationships (5.6) for the above fields are proved by direct calculations.

As in case of general position, here we have three different subcases due to the general structure of invariants in the sequence I_1, I_2, I_3, $\ldots$, they are

(1) the case, when in the infinite series of invariants $I_k(x,y)$ one can find two functionally independent ones;
(2) the case, when invariants $I_k(x,y)$ are functionally dependent, but not all of them are constants;
(3) the case, when all invariants $I_k(x,y)$ are constants.

In the first case algebra of point symmetries of the equation (1.1) is trivial, in the second case this algebra is unidimensional. And in the last case it is two-dimensional. The commutativity conditions for this algebra is written as

$$I_1=-\frac{12}{5}, \qquad\qquad I_2=0. \tag{6.26}$$

When at least one of these conditions (6.26) is broken, the algebra of symmetries is integrable, but not abelian.

7. Second case of intermediate degeneration.

This case is determined by the conditions $P=-F^5=0$, $Q\neq 0$, and $M=0$ in the special coordinates, where the conditions (4.4) are fulfilled. From $M=0$ we have the differential equation $Q_{1.0}=12/5\,Q^2$ for the function $Q(x,y)$. This differential equation is easily solvable:

$$Q=-\frac{5}{12\,x+c(y)}. \tag{7.1}$$

Upon doing the point transformation (4.12) with the proper choice of the function $g(y)$ in it, the field Q from (7.1) can be brought to the form

$$Q=-\frac{5}{12\,x}. \tag{7.2}$$

Then point transformations (4.12) that preserve the form (7.2) of the field Q are determined as follows:

$$\tilde x=\frac{x}{h'(y)^2}, \qquad\qquad \tilde y=h(y). \tag{7.3}$$

By substituting (7.2) into the equation $A = 0$ from (4.4), we derive the differential equation for the function $R(x, y)$:

$$R_{2.0} = -\frac{5}{4}\,\frac{R_{1.0}}{x}. \tag{7.4}$$

This equation (7.4) is also easily solvable. Its general solution has the form

$$R = r(y) + c(y)\,|x|^{-1/4}. \tag{7.5}$$

Now let's consider the case, when $c(y) \neq 0$ in (7.5). Let's do the point transformation (7.3) and take into account that R is a pseudoscalar field of the weight -1 respective to such transformations: $R = (\det T)^{-1}\,\tilde{R}$. At the expense of proper choice of $h(y)$, the field (7.5) can be brought to the following more simple form:

$$R = r(y) + |x|^{-1/4}. \tag{7.6}$$

Point transformations (7.3) that preserve the form (7.6) for R are extremely simple:

$$\tilde{x} = x, \qquad\qquad \tilde{y} = y + \mathrm{const}\,. \tag{7.7}$$

All nonzero coefficients Q, R and S of the equation (1.1) form the scalar fields (scalar invariants) with respect to the transformations (7.7).

Let's substitute (7.2) and (7.6) into the equation $B = 1$ from (4.4). As a result, we get the differential equation for the function $S(x, y)$:

$$S_{2.0} - \frac{5\,S_{1.0}}{4\,x} + \frac{5\,S}{4\,x^2} = 1 - \frac{3\,r(y)}{2\,x\,|x|^{1/4}} - \frac{3}{2\,x\,|x|^{1/2}}. \tag{7.8}$$

General solution for the linear ordinary differential equation (7.8) can be written in the following explicit form:

$$S = \sigma(y)\,|x|^{5/4} - 4\,s(y)\,x + \frac{4}{3}\,|x|^2 - 12\,\frac{r(y)\,|x|^{7/4}}{x} - 4\,\frac{|x|^{3/2}}{x}. \tag{7.9}$$

Let's substitute (7.2), (7.4) and (7.9) into the equations (6.4) and (6.5) for the field of point symmetry. Thereby the equation (6.4) appears to become an identity. The equation (6.5) then is rewritten as follows:

$$\left(-\frac{3}{4}\,\frac{1}{|x|^{1/4}\,x} - \frac{1}{2}\,\frac{r(y)}{x}\right) V + r'(y)\,U = 0. \tag{7.10}$$

Let's differentiate the equation (7.10) with respect to the variable x taking into account the relationships (4.5) and (6.1). As a result, we get the equation

$$-\frac{9}{80}\,\frac{V}{|x|^{1/4}\,x^2} = 0,$$

which leads to the vanishing of one of the components of the vector field of point symmetry: $V = 0$. By substituting $V = 0$ into the equation (7.10), we get $r'(y)\,U = 0$. If the function $r(y)$ is not constant $r'(y) \neq 0$, then $U = 0$ and the algebra of point symmetries of the equation (1.1) is trivial.

In the case, when $r'(y) = 0$, from $V = 0$ we derive that $V_{0.1} = 0$. This lets us write the pair of new compatibility conditions:

$$\frac{\partial V}{\partial x} = V_{1.0}, \qquad\qquad \frac{\partial V_{0.1}}{\partial x} = V_{1.1}. \tag{7.11}$$

First of the relationships (7.11) holds identically, from the second one we derive:

$$\Bigl(\sigma'(y)\,|x|^{5/4} - 4\,s'(y)\,x\Bigr)\;U = 0. \tag{7.12}$$

As an immediate consequence of (7.12) we have the following theorem.

Theorem 7.1. *In the second case of intermediate degeneration the algebra of point symmetries of the equation* (1.1) *is unidimensional if and only if the parameters* $r(y)$, $s(y)$ *and* $\sigma(y)$ *in* (7.4) *and* (7.9) *are identically constant:*

$$r'(y) = 0, \qquad s'(y) = 0, \qquad \sigma'(y) = 0. \tag{7.13}$$

If at least one of the conditions (7.13) *is broken, then the algebra of point symmetries is trivial.*

8. Third case of intermediate degeneration.

This case splits off from the second case of intermediate degeneration by the conditions $c(y) = 0$ and $r(y) \neq 0$ in the formula (7.4). By the proper choice of the function $h(y)$ in the point transformation (7.3), upon applying such transformation, we can get the following relationship:

$$R = 1. \tag{8.1}$$

Due to (8.1), the equation (7.8) is replaced by the following more simple one:

$$S_{2.0} - \frac{5\,S_{1.0}}{4\,x} + \frac{5\,S}{4\,x^2} = 1. \tag{8.2}$$

The general solution for the differential equation (8.2) can be written explicitly:

$$S = \sigma(y)\,|x|^{5/4} - 4\,s(y)\,x + \frac{4}{3}\,|x|^2. \tag{8.3}$$

Let's substitute (7.2), (8.1) and (8.3) into the equations (6.4) and (6.5) for the field of point symmetry. Then the equation (6.4) appears to become an identity. The equation (6.5) reduces to the following relationship

$$-\frac{1}{2}\,\frac{V}{x} = 0,$$

which leads to the vanishing of one of the components of the vector field of point symmetry: $V = 0$. Now we can rewrite the compatibility conditions (7.11), one of them is an identity, another coincides with (7.12).

Theorem 8.1. *In the third case of intermediate degeneration the algebra of point symmetries of the equation* (1.1) *is unidimensional if and only if the parameters* $s(y)$ *and* $\sigma(y)$ *in* (8.3) *are identically constant:*

$$s'(y) = 0, \qquad \sigma'(y) = 0. \tag{8.4}$$

If at least one of the conditions (8.4) *is broken, then the algebra of point symmetries is trivial.*

9. Fourth case of intermediate degeneration.

This case splits off from the second case of intermediate degeneration by the condition of simultaneous vanishing $c(y) = 0$ and $r(y) = 0$ in the formula (7.4). This is equivalent to the vanishing of R:

$$R = 0. \tag{9.1}$$

The equation (8.2) for the function $S(x, y)$ in this case remains unchanged. Therefore S is defined by the formula (8.3). However, in contrast to (8.1), the condition (9.1) specify no subclass in the class of point transformations (7.3). Therefore we can use such transformations for to simplify the formula (8.3). Transformation rule for S by the point transformations (7.3) has the form

$$S = 3\,\frac{x\,h''^2}{h'^2} - 2\,\frac{x\,h'''}{h'} + h'^4\,\tilde{S}. \tag{9.2}$$

From (9.2) we derive the following transformation rules for $\sigma(y)$ and $s(y)$:

$$\begin{aligned} &\sigma(y) = \tilde{\sigma}(\tilde{y})\,|h'(y)|^{3/2},\\ &s(y) = \tilde{s}(\tilde{y})\,h'(y)^2 - \frac{3}{4}\,\frac{h''(y)^2}{h'(y)^2} + \frac{1}{2}\,\frac{h'''(y)}{h'(y)}. \end{aligned} \tag{9.3}$$

From (9.3) we can see that by means of the proper choice of $h(y)$ in (7.3) we can transform $S(x, y)$ to the form

$$S = \sigma(y)\,|x|^{5/4} + \frac{4}{3}\,|x|^2. \tag{9.4}$$

Now we are able to specify the subclass of point transformations (7.3) that preserve the form (9.4) of the field S. Function $h(y)$ for such transformations should satisfy the differential equation $3\,h''^2 = 2\,h'''\,h'$. This is linear-fractional function

$$h(y) = \frac{a\,y + b}{c\,y + d}.$$

In order to calculate the algebra of point symmetries for the equation (1.1) in the present case, let's substitute (7.2), (9.1), and (9.4) into the equations (6.4) and (6.5). Both of them appear to be satisfied identically. This means that for to obtain nontrivial equations for the components of the vector field of point symmetry, in this case, we should return to the equations (2.14) and to the relationships (2.15)–(2.22). From the following two compatibility conditions

$$\frac{\partial U_{0.2}}{\partial y} = U_{0.3}, \qquad\qquad \frac{\partial V_{1.1}}{\partial y} = V_{1.2},$$

we obtain one equation binding V and U. It looks like

$$-3\,\sigma(y)\,V + 4\,x\,\sigma'(y)\,U = 0. \tag{9.5}$$

Other compatibility conditions in the third order of derivatives appears to be fulfilled identically.

Let's add the condition $\sigma(y) \neq 0$ to (9.1). These two conditions, together with $P = 0$, $Q \neq 0$, and (4.4), are the very conditions that specify the fourth case of intermediate degeneration. Using $\sigma(y) \neq 0$, we solve the equation (9.5) with respect to V and then calculate the derivative

$$V_{0.1} = \frac{4}{3}\,\frac{x\,\sigma''}{\sigma}\,U - \frac{20}{9}\,\frac{x\,{\sigma'}^2}{\sigma^2}\,U. \tag{9.6}$$

Now we can consider the following differential consequence from (9.6):

$$\frac{\partial V_{0.1}}{\partial y} = U_{0.2}. \tag{9.7}$$

The relationship (9.7) is written as the equation for U:

$$\left(\sigma''' - 5\,\frac{\sigma''\,\sigma'}{\sigma} + \frac{40}{9}\,\frac{{\sigma'}^3}{\sigma^2}\right) U = 0. \tag{9.8}$$

Other differential consequences add nothing to (9.8) and (9.5).

Theorem 9.1. *In the fourth case of intermediate degeneration the algebra of point symmetries of the equation* (1.1) *is unidimensional if and only if the function* $\sigma(y) \neq 0$ *in* (9.4) *satisfies the differential equation of the form:*

$$\sigma''' - 5\,\frac{\sigma''\,\sigma'}{\sigma} + \frac{40}{9}\,\frac{{\sigma'}^3}{\sigma^2} = 0. \tag{9.9}$$

If the differential equation (9.9) *does not hold, then the algebra of point symmetries is trivial.*

10. Fifth case of intermediate degeneration.

Fifth case splits off from the fourth case of intermediate degeneration by the condition of vanishing $\sigma(y) = 0$ in the formula (9.4). Then the equation (9.5) becomes an identity and no more differential consequences are available. But the differential consequences, which are already available

$$\begin{aligned} &V_{1.0} = \frac{1}{x}\,V, && V_{0.2} = 0,\\ &U_{1.0} = 0, && U_{0.1} = -\frac{1}{2\,x}\,V, \end{aligned} \tag{10.1}$$

can be written as the system of Pfaff equations (2.23). In order to do this, we should construct the vector-column Ψ of the following three functions: V, U, $V_{0.1}$. From $\sigma(y) = 0$ we derive the compatibility of this complete Pfaff system, i. e. the condition $\sigma(y) = 0$ converts to identity the matrix equations (2.24). Compatible system of Pfaff equations (10.1) is easily solvable:

$$V = -4\,a\,x\,y - 4\,b\,x, \qquad U = a\,y^2 + 2\,b\,y + c. \tag{10.2}$$

Its general solution is parameterized by three arbitrary constants a, b, and c. From (10.2) it is easy to get the following characterization for base elements of the algebra of point symmetries of the equation (1.1) for the present case.

1. *Shift of function value*: $x \mapsto x$, $y \mapsto y + t$. Components of corresponding vector field are: $V = 0$, $U = 1$.

2. *Simultaneous blowing up of argument and the function value*: $x \mapsto x\,e^{-2t}$, $y \mapsto y\,e^t$. Components of corresponding vector field are: $V = -2\,x$, $U = y$.

3. *Inversion of the function value with simultaneous blowing up the argument.* Components of vector field are: $V = -2\,x\,y$, $U = y^2/2$. The transformations, forming the one-parametric group, have the form:

$$x \mapsto x\left(1 - \frac{y\,t}{2}\right)^4, \qquad\qquad y \mapsto \frac{2\,y}{2 - y\,t}.$$

Theorem 10.1. *The algebra of point symmetries of the equation* (1.1) *in the fifth case of intermediate degeneration is isomorphic to the matrix algebra* $\mathrm{sl}(2,\mathbb{R})$.

Proof. Arbitrary vector field from the algebra of symmetries is defined by three constants a, b, and c according to the formulas (10.2). Let's consider the map

$$(a, b, c) \mapsto \begin{Vmatrix} b & c \\ a & -b \end{Vmatrix}. \tag{10.3}$$

By means of direct calculations one can prove that this map (10.3) is the very map that establish the isomorphism of the algebra of symmetries of the equation (1.1) and the matrix algebra $\mathrm{sl}(2,\mathbb{R})$. $\square$

11. Sixth case of intermediate degeneration.

In all five previous cases of intermediate degeneration the field Q was nonzero (provided the conditions (4.4) and the equality $P = 0$ are fulfilled). Now let's set $Q = 0$. Then the condition $A = 0$ from (4.4) gives $R_{2.0} = 0$. Therefore

$$R = c(y)\,x + r(y). \tag{11.1}$$

Sixth case of intermediate degeneration is specified by the additional condition $c(y) \neq 0$ in (11.1). The transformation rule for the quantity R under the point transformations (4.12) has the form:

$$R = -\frac{5}{3}\,\frac{h''(y)}{h'(y)} + h'(y)\,\tilde{R}. \tag{11.2}$$

Due to (11.2), at the expense of proper choice of functions $h(y)$ and $g(y)$ in (4.12) we can ensure the relationships $c(y) = 1$ and $r(y) = 0$ upon applying the transformation (4.12). Then for R we have

$$R = x. \tag{11.3}$$

Point transformations (4.12) that preserve the form of the function R in (11.3) form special subclass defined by the formulas

$$\tilde{x} = x, \qquad\qquad \tilde{y} = y + \mathrm{const}\,. \tag{11.4}$$

Both nonzero coefficients R and S in this case are the scalar fields (scalar invariants) respective to the transformations (11.4).

Now let's consider the condition $B = 1$ from (4.4). This condition leads to the differential equation for S:

$$S_{2.0} = 6\,x + 1. \tag{11.5}$$

One can easily write the general solution for (11.5). It has the following form:

$$S = x^3 + \frac{1}{2}\,x^2 + \sigma(y)\,x + s(y). \tag{11.6}$$

In order to calculate the algebra of point symmetries for the equation (1.1), let's substitute $P = 0$ and $Q = 0$ into the equations (4.7) and (4.10). This trivialize both of them and we cannot express the derivatives $U_{0.1}$ and $V_{0.1}$ from them. Therefore we are to consider the differential consequences of the third order derived from (2.14), (4.8), and (4.11). The following four equations

$$\begin{aligned} &\frac{\partial U_{0.2}}{\partial x} = U_{1.2}, &&\frac{\partial U_{1.1}}{\partial y} = U_{1.2},\\ &\frac{\partial V_{1.1}}{\partial x} = V_{2.1}, &&\frac{\partial V_{2.0}}{\partial y} = V_{2.1}\end{aligned}$$

are brought to one equation of the very simple form:

$$U_{0.1} = 0. \tag{11.7}$$

Taking into account (11.7), from next pair of equations of the third order

$$\frac{\partial U_{0.2}}{\partial y} = U_{0.3}, \qquad\qquad \frac{\partial V_{1.1}}{\partial y} = V_{1.2},$$

we derive one more equation that determine the derivative $V_{0.1}$:

$$V_{0.1} = \left(-\frac{14}{15}\,x - \frac{5}{9}\right)\,V - \frac{5}{9}\,\sigma'(y)\,U. \tag{11.8}$$

Other compatibility equations of the third order add nothing new. But nevertheless, using (11.7) and (11.8), we can come back to the second order compatibility conditions. Now they are reduced to the pair of relationships, one of which is $V = 0$. Another one has the form:

$$\left(s'(y) - \frac{4}{27}\,\sigma'(y)\,x + \frac{25}{81}\,\sigma'(y) - \frac{5}{9}\,\sigma''(y)\right) U = 0. \tag{11.9}$$

From $V = 0$ and from (11.9) we can derive the following theorem.

Theorem 11.1. *In the sixth case of intermediate degeneration the algebra of point symmetries of the equation* (1.1) *is unidimensional if and only if the parameters* $s(y)$ *and* $\sigma(y)$ *in* (11.6) *are identically constant:*

$$s'(y) = 0, \qquad\qquad \sigma'(y) = 0. \tag{11.10}$$

If at least one of the conditions (11.10) *is broken, then the algebra of point symmetries is trivial.*

12. Seventh case of intermediate degeneration.

This case splits off from sixth case of intermediate degeneration by the condition $c(y) = 0$ in the formula (11.1). Using the transformation rule (11.2), in this case, we can provide the vanishing of the field R:

$$R = 0. \tag{12.1}$$

Point transformations of the form (4.12) that preserve the condition (12.1) are defined by the following formulas:

$$\tilde{x} = \frac{x}{a^2} + g(y), \qquad\qquad \tilde{y} = a\,y + b. \tag{12.2}$$

From the condition $B = 1$ in (4.4), we get the differential equation $S_{2.0} = 1$ for the field S. General solution of this trivial equation has the form:

$$S = \frac{1}{2}\,x^2 + \sigma(y)\,x + s(y). \tag{12.3}$$

Point transformations (12.2) transform the field S according to the following rule

$$S = a^2\,g''(y) + a^4\,\tilde{S}. \tag{12.4}$$

From (12.4) and from (12.2) one easily derive the transformation rules for the coefficients $\sigma(y)$ and $s(y)$ in (12.3). They have the form:

$$\begin{aligned} &\sigma(y) = a^2\,\tilde{\sigma}(\tilde{y}) + a^2\,g(y),\\ &s(y) = a^4\,\tilde{s}(\tilde{y}) + a^2\,g''(y) + a^4\,\tilde{\sigma}(\tilde{y})\,g(y) + \frac{1}{2}\,a^4\,g(y)^2. \end{aligned} \tag{12.5}$$

Now, from (12.5), we see that, using the arbitrariness in the choice of local coordinates, defined by the point transformations (12.2), we always can vanish $\sigma(y)$ in (12.3). For S from $\sigma(y) = 0$ we get

$$S = \frac{1}{2}\,x^2 + s(y). \tag{12.6}$$

Point transformations (12.2) that preserve the form of field S in (12.6) form the subclass that doesn't contain the functional parameters:

$$\tilde{x} = \frac{x}{a^2}, \qquad \tilde{y} = a\,y + b. \tag{12.7}$$

The field S is a pseudoscalar field of the weight -4 respective to the transformations (12.7). But we can easily construct the scalar field I:

$$I = \frac{S}{x^2}.$$

Here, as in the previous case, the equations (4.7) and (4.10) are identically zero due to $P = 0$ and $Q = 0$. In order to find the algebra of point symmetries, we should consider the differential consequences of higher order. The following two compatibility equations of the third order

$$\frac{\partial U_{0.2}}{\partial y} = U_{0.3}, \qquad \frac{\partial V_{1.1}}{\partial y} = U_{1.2}$$

are reduced to one relationship, defining the derivative $U_{0.1}$:

$$U_{0.1} = -\frac{1}{2\,x}\,V. \tag{12.8}$$

This relationship (12.8) gives rise to the new compatibility conditions of the second order. Two of them

$$\frac{\partial U_{0.1}}{\partial y} = U_{0.2}, \qquad \frac{\partial V_{1.0}}{\partial y} = U_{1.1}$$

are brought to the relationship that leads to the vanishing of the derivative $V_{0.1}$:

$$V_{0.1} = 0. \tag{12.9}$$

From (12.9) we derive the equation $V_{0.2} = 0$. In explicit form this equation is written as:

$$\frac{2}{x}\,s(y)\,V - s'(y)\,U = 0. \tag{12.10}$$

When $s(y) = 0$, the equation (12.10) is fulfilled identically and no more equations are available. Therefore we have the following theorem.

Theorem 12.1. *In the seventh case of intermediate degeneration the algebra of point transformations of the equation* (1.1) *is two-dimensional if and only if the function $s(y)$ in* (12.6) *is identically zero. This algebra is integrable but it is not Abelian.*

Now let's suppose that $s(y) \neq 0$. In this case the equation (12.10) can be solved with respect to V and we can derive two differential consequences of the first order from it. One of them has the form:

$$\left(\frac{4\,s''(y)}{s(y)} - \frac{5\,s'(y)^2}{s(y)^2}\right) U = 0.$$

Theorem 12.2. *In the seventh case of intermediate degeneration the algebra of point transformations of the equation* (1.1) *is unidimensional if and only if the function $s(y)$ in* (12.6) *is the solution of the following differential equation:*

$$4\,s''(y) - \frac{5\,s'(y)^2}{s(y)} = 0. \tag{12.11}$$

If the differential equation (12.11) *does not hold, then the algebra of point symmetries is trivial.*

13. Appendix A. Systems of Pfaff equations.

Let's consider the system of n functions: $u^1(x,y),\ldots,u^n(x,y)$. The following two systems of equations form *the complete system of Pfaff equations*:

$$\left\{\begin{array}{l} u^1_{1.0}=U^1_{1.0}(u^1,\ldots,u^n,x,y),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u^n_{1.0}=U^n_{1.0}(u^1,\ldots,u^n,x,y),\end{array}\right. \qquad \left\{\begin{array}{l} u^1_{0.1}=U^1_{0.1}(u^1,\ldots,u^n,x,y),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u^n_{0.1}=U^n_{0.1}(u^1,\ldots,u^n,x,y).\end{array}\right. \tag{13.1}$$

Right hand sides of these equations are the functions of n dependent variables $u^1,\ldots,u^n$, and two independent variables x and y. Differentiation of such function with respect to x and y is given by the following differential operators:

$$\begin{aligned} D_x&=\frac{\partial}{\partial x}+\sum_{i=1}^{n}U^i_{1.0}(u^1,\ldots,u^n,x,y)\,\frac{\partial}{\partial u^i},\\ D_y&=\frac{\partial}{\partial y}+\sum_{i=1}^{n}U^i_{0.1}(u^1,\ldots,u^n,x,y)\,\frac{\partial}{\partial u^i}.\end{aligned}\tag{13.2}$$

From (13.1), as a result of successive differentiations, we get the series of formulas for the derivatives $u^i_{p.q}$ of the arbitrary order:

$$u^i_{p.q}=U^i_{p.q}(u^1,\ldots,u^n,x,y).\tag{13.3}$$

Right hand sides of the equations (13.3) are bound with each other by the recurrent relationships of the form:

$$U^i_{p+1.q}=D_xU^i_{p.q},\qquad\qquad U^i_{p.q+1}=D_yU^i_{p.q}.\tag{13.4}$$

Let's represent the relationships (13.4) by means of diagrams:

$$\begin{tikzcd} {} & {} & {} & \\ U^i_{2.0} \arrow[u] \arrow[r] & U^i_{2.1} \arrow[u] \arrow[r] & U^i_{2.2} \arrow[u] \arrow[r] & \ldots \\ U^i_{1.0} \arrow[u] \arrow[r] & U^i_{1.1} \arrow[u] \arrow[r] & U^i_{1.2} \arrow[u] \arrow[r] & \ldots \\ & U^i_{0.1} \arrow[u] \arrow[r] & U^i_{0.2} \arrow[u] \arrow[r] & \ldots \end{tikzcd}\tag{13.5}$$

Vertical arrows in (13.5) represent the action of the operator D_x, while horizontal ones represent the action of the operator D_y from (13.2). Recurrent relationships (13.4) shown in the diagrams (13.5) are redundant. They define uniquely only the side vertices in the diagrams (13.5), which have only one incoming arrow. Other vertices with two incoming arrows give rise to the compatibility conditions. Thus in the vertex $U^i_{p+1.q+1}$ we have the equation

$$D_xU^i_{p.q+1}=D_yU^i_{p+1.q}.\tag{13.6}$$

Lemma 13.1. *The set of compatibility conditions* (13.6) *written for the vertices* $U^i_{1.1}$ *with* $i = 1, \dots, n$ *is equivalent to the permutability of the operators* D_x *and* D_y *defined by* (13.2).

Proof. Let's calculate the commutator $[D_x,\, D_y]$ of the operators D_x and D_y in explicit form. This is the following differential operator of the first order:

$$[D_x,\, D_y] = \sum_{i=1}^{n} \left(D_x U^i_{0.1} - D_y U^i_{1.0}\right) \frac{\partial}{\partial u^i}. \tag{13.7}$$

The relationships (13.6) for the vertices $U^i_{1.1}$ are written as: $D_x U^i_{0.1} = D_y U^i_{1.0}$. It is easy to see that they hold for all $i = 1, \dots, n$ if and only if the commutator (13.7) is equal to zero. Lemma is proved. $\square$

Lemma 13.2. *If* $[D_x, D_y] = 0$, *then the compatibility conditions* (13.6) *are fulfilled for all vertices* $U^i_{p+1.q+1}$ *in the diagrams* (13.5).

Proof. For the vertex $U^i_{p.q}$ let's consider the number $r = p + q$. We shall call it *the order* of this vertex. We shall prove lemma by induction in r. The result of lemma 13.1 forms the base for such induction for $r = 2$.

Suppose that lemma is proved for all vertices of the order not greater than $p + q + 1$. The vertex $U^i_{p+1.q+1}$ of the order $r = p + q + 2$ is in the following rectangular part of the diagram (13.5):

$$\begin{array}{ccc} U_{p+1.q} & \longrightarrow & U_{p+1.q+1} \\ \uparrow & & \uparrow \\ U_{p.q} & \longrightarrow & U_{p.q+1} \end{array}$$

Due to inductive hypothesis, the compatibility relationship (13.6) for $U^i_{p+1.q+1}$ can be rewritten as:

$$D_x U^i_{p.q+1} - D_y U^i_{p+1.q} = D_x Dy U^i_{p.q} - D_y D_x U^i_{p.q} = [D_x,\, D_y] U^i_{p.q} = 0.$$

Now its clear that this relationship can be derived from $[D_x,\, D_y] = 0$. Lemma is proved. $\square$

Theorem 13.1. *Complete system of Pfaff equations* (13.1) *is compatible if and only if the corresponding operators* (13.2) *are commutating.*

Theorem 13.1 is the consequence of lemmas 13.1 and 13.2. It gives us an effective tool for checking the compatibility of complete Pfaff equations.

Theorem 13.2. *Each complete compatible system of Pfaff equations* (13.1) *is locally solvable.*

Proof. Suppose that the condition $[D_x,\, D_y] = 0$ for the system of equations (13.1) is fulfilled. In order to prove the local solvability for these equations, let's state the Cauchy problem with the following initial data for them:

$$u^i\Big|_{\substack{x=0\\y=0}} = \alpha^i_0. \tag{13.8}$$

For to solve this Cauchy problem (13.8), note that each system of equations in (13.1) can be treated as a system of ordinary differential equations in x and in y separately. For the second system in (13.1) we consider the standard Cauchy problem

$$u^i\Big|_{y=0} = \alpha^i(x). \tag{13.9}$$

For the initial value functions α^i in (13.9) from the first system (13.1) we get

$$\begin{cases} (\alpha^1)'_x = U^1_{0.1}(\alpha^1, \ldots, \alpha^n, x, 0), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\alpha^n)'_x = U^n_{0.1}(\alpha^1, \ldots .\alpha^n, x, 0), \end{cases} \tag{13.10}$$

It is easy to see that (13.10) is a system of ordinary differential equations in x. For this system, from (13.8), we derive:

$$\alpha^i\Big|_{x=0} = \alpha^i_0. \tag{13.11}$$

It is clear that (13.11) is a standard Cauchy problem for (13.10).

As a result of successive solution of two standard Cauchy problems (13.11) and (13.9), we get the set of functions $u^1(x,y), \ldots, u^n(x,y)$ which satisfies the second system of Pfaff equations in (13.1). Now we are only to prove that these functions satisfy the first system (13.1) too. In order to do this, let's substitute $u^1(x,y), \ldots, u^n(x,y)$ into the first system

$$
\begin{aligned}
&u^1_{1.0} = U^1_{1.0}(u^1, \ldots, u^n, x, y) + \delta^1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&u^n_{1.0} = U^n_{1.0}(u^1, \ldots, u^n, x, y) + \delta^n,
\end{aligned}
\tag{13.12}
$$

and calculate the residuals $\delta^i(x,y)$. Then we differentiate the relationships (13.12) with respect to y. For the residuals $\delta^i(x,y)$ this gives:

$$
\delta^i_{0.1} = u^i_{1.1} - \left(\frac{\partial U^i_{1.0}}{\partial y} + \sum_{j=1}^{n} u^j_{0.1}\,\frac{\partial U^i_{1.0}}{\partial u^j}\right) = D_x U^i_{0.1} - D_y U^i_{1.0} - \sum_{j=1}^{n} \frac{\partial U^i_{1.0}}{\partial u^j}\,\delta^j.
$$

But $D_x U^i_{0.1} = D_y U^i_{1.0}$, which is the consequence of the compatibility of the equations (13.1). Therefore the residuals δ^i satisfy the system of homogeneous differential equations in y of the following form:

$$
\delta^i_{0.1} = -\sum_{j=1}^{n} \frac{\partial U^i_{1.0}}{\partial u^j}\,\delta^j.
\tag{13.13}
$$

From (13.10) and (13.12) we derive zero initial data for the equations (13.13):

$$
\delta^i\Bigr|_{y=0} = 0.
\tag{13.14}
$$

From (13.13) and (13.14) we obtain the identical vanishing of the residuals $\delta^i(x,y)$. Theorem is proved. $\square$

From the above proof of the theorem 13.2 we see that the solution of the Cauchy problem (13.8) for the compatible system of Pfaff equations is unique. Therefore the general solution of such system of equations is n-parametric family of functions parameterized by constants a^i_0 in (13.8).

14. Appendix B. Pfaff equations with restrictions.

Let's consider the Pfaff equations (13.1). Right hand sides of these equations contain the independent variables x and y in explicit form. Such Pfaff equations are called *nonholonomic*. But each nonholonomic system of Pfaff equations can be transformed to the *holonomic* one by increasing its dimension $n \to n+2$. Let's add two dependent variables $u^{n+1} = x$ and $u^{n+2} = y$ and define four new functions

$$
\begin{aligned}
&U^{n+1}_{1.0}(u^1, \ldots, u^{n+2}) = 1, &&U^{n+1}_{0.1}(u^1, \ldots, u^{n+2}) = 0,\\
&U^{n+2}_{1.0}(u^1, \ldots, u^{n+2}) = 0, &&U^{n+2}_{1.0}(u^1, \ldots, u^{n+2}) = 1.
\end{aligned}
\tag{14.1}
$$

These functions (14.1) let us to add two new equations to each system of Pfaff equations in (13.1). On doing this, we get complete holonomic system of Pfaff equations of the dimension $n+2$. This operation is called *holonomic expansion* of the equations (13.1). It is easy to check that the operation of holonomic expansion preserves the compatibility of Pfaff systems.

Operations, that diminish the dimension of Pfaff equations are called *restrictions*. Regular way of doing this is connected with adding some restricting equations to the system. Let's consider the complete holonomic system of Pfaff equations:

$$\left\{\begin{array}{l} u^1_{1.0} = U^1_{1.0}(u^1, \dots, u^n),\\ \dots\dots\dots\dots\dots\dots\dots\\ u^n_{1.0} = U^n_{1.0}(u^1, \dots, u^n), \end{array}\right. \qquad \left\{\begin{array}{l} u^1_{0.1} = U^1_{0.1}(u^1, \dots, u^n),\\ \dots\dots\dots\dots\dots\dots\dots\\ u^n_{0.1} = U^n_{0.1}(u^1, \dots, u^n). \end{array}\right. \tag{14.2}$$

Then consider the following system of functional equations:

$$\left\{\begin{array}{l} I^1(u^1, \dots, u^n) = 0,\\ \dots\dots\dots\dots\dots\dots\\ I^k(u^1, \dots, u^n) = 0. \end{array}\right. \tag{14.3}$$

This system of equations (14.3) is assumed to be *regular*. Such system defines $(n-k)$-dimensional submanifold M in the space $\mathbb{R}^n$ of variables $u^1, \dots, u^n$. Each solution of Pfaff equations (14.2) defines two-parametric subset in the same space $\mathbb{R}^n$. If $S \subset M$, then we have the following relationships

$$\sum_{q=1}^{n} \frac{\partial I^k}{\partial u^q}\, U^q_{1.0} = 0, \qquad \sum_{q=1}^{n} \frac{\partial I^k}{\partial u^q}\, U^q_{0.1} = 0. \tag{14.4}$$

The relationships (14.4) have the same structure as the equations (14.3). We say that the restrictions (14.3) are *consistent* with Pfaff equations (14.2), if the relationships (14.4) are functional consequences of the equations (14.3), i. e. if they hold identically on the manifold M.

Definition 14.1. Differential Pfaff equations (14.2) equipped with consistent functional equations (14.3) are called *restricted* Pfaff equations.

If the condition of consistence is not fulfilled, we can add the equations (14.4) to the system (14.3). Then we extract the maximal functionally independent subsystem of equations in this enlarged system. It is clear that such subsystem will be consistent with (14.2). Therefore we can always consider only the consistent systems of restrictions (14.3).

Now let's remember the operators D_x and D_y from (13.2). Here they can be treated as the vector fields in $\mathbb{R}^n$:

$$\begin{aligned} D_x &= \sum_{i=1}^{n} U^i_{1.0}(u^1, \dots, u^n)\, \frac{\partial}{\partial u^i},\\ D_y &= \sum_{i=1}^{n} U^i_{0.1}(u^1, \dots, u^n)\, \frac{\partial}{\partial u^i}. \end{aligned} \tag{14.5}$$

The consistence of (14.3) and (14.2) means that vector fields (14.5) are tangent to the submanifold M. It is known that tangent fields can be restricted to submanifold.

Let $v^1, \dots, v^{n-k}$ be the system of local coordinates on M. Then the restrictions of the vector field (14.5) on M can be represented as:

$$
\begin{aligned}
&\tilde{D}_x = \sum_{i=1}^{n-k} V^i_{1.0}(v^1, \dots, v^{n-k}) \frac{\partial}{\partial v^i},\\
&\tilde{D}_y = \sum_{i=1}^{n-k} V^i_{0.1}(v^1, \dots, v^{n-k}) \frac{\partial}{\partial v^i}.
\end{aligned}
\tag{14.6}
$$

These vector fields (14.6) can define some $(n-k)$-dimensional system of Pfaff equations on M

$$
\begin{cases}
v^1_{1.0} = V^1_{1.0}(v^1, \dots, v^{n-k}),\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
v^{n-k}_{1.0} = V^{n-k}_{1.0}(v^1, \dots, u^{n-k}),
\end{cases}
\qquad
\begin{cases}
v^1_{0.1} = V^1_{0.1}(v^1, \dots, v^{n-k}),\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
v^{n-k}_{0.1} = V^{n-k}_{0.1}(v^1, \dots, v^{n-k}).
\end{cases}
\tag{14.7}
$$

It is obvious that each solution the equations (14.7) defines some solution for (14.2), which is the solution for (14.3) too. And conversely, each solution of the restricted Pfaff equations (14.2) and (14.3) gives some solution for (14.7). Therefore the system of Pfaff equations (14.7) is called *the restriction* of the equations (14.2) due to (14.3).

The following two facts are well known in differential geometry: commutator of two vector fields tangent to the submanifold M is also tangent to M; the restriction of commutator for two tangent vector fields coincides with the commutator of their restrictions. These facts make natural the following definition.

Definition 14.2. Complete system of restricted Pfaff equations (14.2) with restrictions (14.3) is called *compatible*, if the commutator of corresponding differential operators (14.5) vanishes at any point of submanifold M defined by (14.3).

It is clear that the compatibility of the Pfaff equations (14.2) with restrictions (14.3) in the sense of definition 14.2 is equivalent to the compatibility of its restriction (14.7) in the sense of theorem 13.1. Therefore from the results of previous section we derive the following theorem.

Theorem 14.1. *Each complete compatible system of Pfaff equations* (14.2) *with restrictions* (14.3) *is locally solvable and in some neighborhood of any point* (x, y) *it possess* $(n-k)$*-parametric set of solutions.*

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