ON THE GEOMETRY OF POINT-EXPANSIONS FOR CERTAIN CLASS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

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Abstract. Second order ordinary differential equations of the form $y'' = P(x, y) + 4\,Q(x, y)\,y' + 6\,R(x, y)\,{y'}^2 + 4\,S(x, y)\,{y'}^3 + L(x, y)\,{y'}^4$ are considered and their point-expansions are constructed. Geometrical structures connected with these expansions are described.

1. Introduction.

Class of ordinary differential equations of the second order with the third order polynomials in y' in right hand side

$$y'' = P(x, y) + 3\,Q(x, y)\,y' + 3\,R(x, y)\,{y'}^2 + S(x, y)\,{y'}^3 \tag{1.1}$$

is closed with respect to the point transformations of the form

$$\begin{cases} \tilde{x} = \tilde{x}(x, y), \\ \tilde{y} = \tilde{y}(x, y). \end{cases} \tag{1.2}$$

This means that on applying this change of variables to (1.1) we shall get another equation of the same form

$$\tilde{y}'' = \tilde{P}(\tilde{x}, \tilde{y}) + 3\,\tilde{Q}(\tilde{x}, \tilde{y})\,\tilde{y}' + 3\,\tilde{R}(\tilde{x}, \tilde{y})\,{\tilde{y}'}^2 + \tilde{S}(\tilde{x}, \tilde{y})\,{\tilde{y}'}^3. \tag{1.3}$$

For two particular equations (1.1) and (1.3) the problem of the existence of the point transformation (1.2) transforming one of them into another is known as *the problem equivalence*. Different aspects of this problem were studied during the long time (see [1-22]). In paper [23] one can find the scheme of point-classification for the equations of the form (1.1) and the complete list of cases and subcases arising in this scheme.

Let's increase by 1 the degree of polynomial in right hand side of (1.1). Then we get the class of equations

$$y'' = P(x, y) + 4\,Q(x, y)\,y' + 6\,R(x, y)\,{y'}^2 + 4\,S(x, y)\,{y'}^3 + L(x, y)\,{y'}^4, \tag{1.4}$$

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which is not closed with respect to the point transformations (1.2). Change of variables (1.2) gives rise to the denominator in (1.4):

$$y'' = \frac{P(x,y) + 4\,Q(x,y)\,y' + 6\,R(x,y)\,{y'}^2 + 4\,S(x,y)\,{y'}^3 + L(x,y)\,{y'}^4}{Y(x,y) - X(x,y)\,y'}. \tag{1.5}$$

Class of the equations (1.5) is closed with respect to the point transformations (1.2). Let's call it the point-expansion for the class of the equations (1.4). The main purpose of this paper consists in describing the geometrical structures connected with these equations.

2. Point transformations.

Change of variables (1.2) in differential equations can be treated as the change of some curvilinear coordinates on the plane for others. This is the way for geometrization of different problems associated with such transformations. We shall take the point transformations (1.2) to be regular and we denote by T and S direct and inverse matrices of Jacoby for them:

$$S = \begin{Vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{Vmatrix}, \qquad T = \begin{Vmatrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{Vmatrix}. \tag{2.1}$$

According to the paper [23] by means of double indices in (2.1) and in what follows we denote partial derivatives. Thus for the function f of two arguments by $f_{p.q}$ we denote the differentiation p times with respect to the first argument and q times with respect to the second one.

Here is the formula for transforming the first derivative y' under the point transformations (1.2):

$$y' = \frac{y_{1.0} + y_{0.1}\,\tilde{y}'}{x_{1.0} + x_{0.1}\,\tilde{y}'}. \tag{2.2}$$

Appropriate formula for the second derivative y'' is the following one:

$$\begin{aligned} y'' = {} & \frac{(x_{1.0} + x_{0.1}\,\tilde{y}')(y_{2.0} + 2\,y_{1.1}\,\tilde{y}' + y_{0.2}\,(\tilde{y}')^2 + y_{0.1}\,\tilde{y}'')}{(x_{1.0} + x_{0.1}\,\tilde{y}')^3} - \\ & - \frac{(y_{1.0} + y_{0.1}\,\tilde{y}')(x_{2.0} + 2\,x_{1.1}\,\tilde{y}' + x_{0.2}\,(\tilde{y}')^2 + x_{0.1}\,\tilde{y}'')}{(x_{1.0} + x_{0.1}\,\tilde{y}')^3}. \end{aligned} \tag{2.3}$$

Substituting (2.2) and (2.3) into (1.5) we can get the transformation rules for the coefficients of the equation (1.5). First let's consider the transformation rules for the parameters X and Y in denominator:

$$\begin{aligned} X &= u\,\left(S^1_1\,\tilde{X} + S^1_2\,\tilde{Y}\right), \\ Y &= u\,\left(S^2_1\,\tilde{X} + S^2_2\,\tilde{Y}\right). \end{aligned} \tag{2.4}$$

Here $u = u(x, y)$ is an arbitrary function of two variables. It is present in (2.4) since numerator and denominator of a fraction can be multiplied by the same factor $u(x, y) \neq 0$ without change of its value. This doesn't change the form of dependence on y' too. For the sake of certainty let's take $u = 1$. Then the transformation rules for X and Y can be written in matrix form using the matrix S from (2.1):

$$\begin{Vmatrix} X \\ Y \end{Vmatrix} = \begin{Vmatrix} S^1_1 & S^1_2 \\ S^2_1 & S^2_2 \end{Vmatrix} \cdot \begin{Vmatrix} \tilde{X} \\ \tilde{Y} \end{Vmatrix}. \tag{2.5}$$

The rule (2.5) is exactly the rule of transformations of the components of a vector under the change of local curvilinear coordinates on the plane (for more details see [24]). It enables us to construct the vector field $\boldsymbol{\alpha}$ with the following components:

$$\alpha^1 = X, \qquad\qquad \alpha^2 = Y. \tag{2.6}$$

It's clear that $\boldsymbol{\alpha} \neq 0$ since if both components of this field vanish simultaneously, then this leads to the vanishing of the denominator in (1.5).

Apart from ordinary vectorial and tensorial fields, in the theory of point transformations for the equations (1.1) and (1.5) we have the pseudotensorial fields with certain weight (see [25] and [23]).

Definition 2.1. Pseudotensorial field of the type (r, s) and weight m is the multidimensional array $F^{i_1 \dots\, i_r}_{j_1 \dots\, j_s}$ which is transformed as

$$F^{i_1 \dots\, i_r}_{j_1 \dots\, j_s} = (\det T)^m \sum_{\substack{p_1 \dots p_r \\ q_1 \dots q_s}} S^{i_1}_{p_1} \dots S^{i_r}_{p_r}\, T^{q_1}_{j_1} \dots T^{q_s}_{j_s}\, \tilde{F}^{p_1 \dots\, p_r}_{q_1 \dots\, q_s}$$

under the point changes of variables of the form (1.2).

Let's substitute $y' = z$ into the right hand side of (1.5) and let's consider the resulting fraction as the rational function of the variable z:

$$f(z) = \frac{P + 4\,Q\,z + 6\,R\,z^2 + 4\,S\,z^3 + L\,z^4}{Y - X\,z}. \tag{2.7}$$

Function (2.7) has a pole at the point $z_0 = Y/X$. Let's calculate the residue of this function at z_0 and denote by $\Omega = \Omega(x, y)$ the following quantity:

$$\Omega = -X^5 \operatorname*{Res}_{z=z_0} f(z). \tag{2.8}$$

It's easy to get an explicit expression for the quantity Ω defined in (2.8):

$$\Omega = P\,X^4 + 4\,Q\,X^3\,Y + 6\,R\,X^2\,Y^2 + 4\,S\,X\,Y^3 + L\,Y^4. \tag{2.9}$$

It's easy also to check that the quantity (2.9) is transformed as the pseudoscalar field of the weight -2 under the point transformations (1.2):

$$\Omega = (\det T)^{-2}\,\tilde{\Omega}. \tag{2.10}$$

The relationship (2.10) can be derived by means of direct calculations on the base of transformation rules for the coefficients of the equation (1.5). As for these rules themselves, we shall not write them here since they are rather huge in explicit form.

3. Special coordinates.

For the further analysis of the equations (1.5) we shall use the following well-known theorem on the straightening of the vector field (see [26] or [27]).

Theorem 3.1. *For any nonzero vector field $\boldsymbol{\alpha}$ on the plane with the components α^1 and α^2 in coordinates x and y one can find the point transformation* (1.2) *such that in coordinates $\tilde{x}$ and $\tilde{y}$ this field $\boldsymbol{\alpha}$ is expressed by $\tilde{\alpha}^1 = 0$ and $\tilde{\alpha}^2 = 1$.*

Applying this theorem to the equation (1.5) we obtain the following result.

Corollary. *Any equation* (1.5) *can be transformed to the form* (1.4) *by means of point change of variables of the form* (1.2).

Let's choose special coordinates x and y, in which the vector field $\boldsymbol{\alpha}$ has the unit components:

$$\tilde{\alpha}^1 = 0, \qquad \tilde{\alpha}^2 = 1. \tag{3.1}$$

Now let's consider the point transformations that preserve the condition (3.1). They form special subclass among general point transformations of the form (1.2) given by the following relationships:

$$\tilde{x} = h(x), \qquad \tilde{y} = y + g(x). \tag{3.2}$$

By differentiating these relationships (3.2) we find the components of transition matrices S and T from (2.1):

$$S = \frac{1}{h'} \begin{Vmatrix} 1 & 0 \\ -g' & h' \end{Vmatrix}, \qquad T = \begin{Vmatrix} h' & 0 \\ g' & 1 \end{Vmatrix}. \tag{3.3}$$

Since these matrices are not degenerate, we get $\det T = h'(x) \neq 0$.

In the special coordinates, which were chosen above, the equation (1.5) has the form (1.4). This is due to (3.1). Let's consider the transformation rules for the parameters L and S in (1.4) under the point transformations (3.2), which preserve the condition (3.1):

$$L = \frac{1}{h'^2}\,\tilde{L}, \qquad S = \frac{g'}{h'^2}\,\tilde{L} + \frac{1}{h'}\,\tilde{S}. \tag{3.4}$$

First of the relationships (3.4) means that the parameter L is transformed as pseudoscalar field of the weight -2. This is not surprising since in case, when the conditions (3.1) hold, the formula (2.9) for the field Ω gives $\Omega = L$.

Conclusion. Parameter L in special coordinates defines the pseudoscalar field of the weight -2 that can be extended as the field Ω for arbitrary coordinates.

Let's take one more vector field $\boldsymbol{\psi}$ other than vector field $\boldsymbol{\alpha}$. Its components in special coordinates are given by the following relationships:

$$\psi^1 = L, \qquad \psi^2 = -S. \tag{3.5}$$

Then we can represent the formulas (3.4) in matrix form:

$$\left\|\begin{matrix}\psi^1\\ \psi^2\end{matrix}\right\| = \frac{1}{h'}\left\|\begin{matrix}S^1_1 & S^1_2\\ S^2_1 & S^2_2\end{matrix}\right\| \cdot \left\|\begin{matrix}\tilde\psi^1\\ \tilde\psi^2\end{matrix}\right\|. \tag{3.6}$$

From (3.6) we see that the field $\boldsymbol{\psi}$ defined by (3.5) in special coordinates is the pseudovectorial field of the weight -1. Note that the condition of noncollinearity of the fields $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$ coincides with $L \neq 0$ in special coordinates, and with $\Omega \neq 0$ in arbitrary ones. For the equations (1.4) and (1.5) this condition surely holds since had it been broken, this would reduce the equations (1.4) and (1.5) to the form (1.1).

Having defined the field $\boldsymbol{\psi}$ in special coordinates it's natural to calculate its components in arbitrary coordinates. Let's denote them by M and N:

$$\psi^1 = M, \qquad\qquad \psi^2 = N. \tag{3.7}$$

Then M and N are defined by the following two formulas:

$$\begin{aligned} M = Q\,X^3 + 3\,R\,X^2\,Y + 3\,S\,X\,Y^2 + L\,Y^3 +\\ + \frac{X}{4}\left(X^2\,Y_{1.0} - X\,X_{1.0}\,Y + X\,Y\,Y_{0.1} - X_{0.1}\,Y^2\right).\end{aligned} \tag{3.8}$$

$$\begin{aligned} N = -P\,X^3 - 3\,Q\,X^2\,Y - 3\,R\,X\,Y^2 - S\,Y^3 +\\ + \frac{Y}{4}\left(X^2\,Y_{1.0} - X\,X_{1.0}\,Y + X\,Y\,Y_{0.1} - X_{0.1}\,Y^2\right).\end{aligned} \tag{3.9}$$

In order to prove the formulas (3.8) and (3.9) first let's check that in special coordinates they give $M = L$ and $N = -S$. Then we should verify the following transformation rules for M and N given by (3.8) and (3.9):

$$\left\|\begin{matrix}M\\ N\end{matrix}\right\| = (\det T)^{-1}\left\|\begin{matrix}S^1_1 & S^1_2\\ S^2_1 & S^2_2\end{matrix}\right\| \cdot \left\|\begin{matrix}\tilde M\\ \tilde N\end{matrix}\right\|. \tag{3.10}$$

This can be done by direct calculations based on the transformation rules for the coefficients of the equation (1.5).

Vector field $\boldsymbol{\alpha}$, pseudovectorial field $\boldsymbol{\psi}$, and pseudoscalar field Ω are bound with the following relationship:

$$\Omega = \sum_{i=0}^{2}\sum_{j=0}^{2} d_{ij}\,\psi^i\,\alpha^j. \tag{3.11}$$

Here d_{ij} is the unit skew-symmetric matrix 2×2:

$$d_{ij} = \left\|\begin{matrix}0 & 1\\ -1 & 0\end{matrix}\right\|. \tag{3.12}$$

Matrix (3.12) defines twice-covariant pseudotensorial field of the weight -1.

After explicit calculation of the sums in right hand side of (3.11) we may rewrite this formula as follows:

$$\Omega = M\,Y - N\,X. \tag{3.13}$$

Now the relationship (3.13) can be easily derived from (2.9), (3.8), and (3.9) by direct calculations.

4. Associated equation.

Now let's use the components of the fields $\boldsymbol{\alpha}$ and $\boldsymbol{\psi}$, and the pseudoscalar field Ω for to construct the following fraction:

$$k(z)=\frac{(N-M\,z)^4}{\Omega^3(Y-X\,z)}. \tag{4.1}$$

Both functions, $k(z)$ from (4.1) and $f(z)$ from (2.7), have simple pole at the same point $z_0=Y/X$. Due to the relationships (3.13) their residues are equal. Therefore the difference of these two functions is a polynomial:

$$f(z)-k(z)=P^*(x,y)+3\,Q^*(x,y)\,z+3\,R^*(x,y)\,z^2+S^*(x,y)\,z^3. \tag{4.2}$$

It's easy to calculate the coefficients of the polynomial (4.2) in special coordinates, where $X=0$ and $Y=1$:

$$\begin{aligned}
&P^*=P-\frac{S^4}{L^3}, &&Q^*=\frac{4Q}{3}-\frac{4S^3}{L^2},\\
&R^*=2R-\frac{2S^2}{L}, &&S^*=0.
\end{aligned} \tag{4.3}$$

As for the coefficients P^*, Q^*, R^*, S^* in arbitrary coordinates, they can be expressed trough P, Q, R, S, L, X, Y. But these expressions are huge enough. Instead of writing them in explicit form, we shall prove the following theorem.

Theorem 4.2. *Coefficients of the polynomial* (4.2) *are defined uniquely by the differential equation* (1.5).

Proof. Using (4.1) and (4.2) we can calculate P^*, Q^*, R^*, S^* trough P, Q, R, S, L, X, and Y. However, the parameters P, Q, R, S, L, X, Y are not determined uniquely by the equation (1.5). Their choice admits gauge arbitrariness of the form

$$\begin{aligned}
&P\to\varphi(x,y)\,P, &&Q\to\varphi(x,y)\,Q,\\
&R\to\varphi(x,y)\,R, &&S\to\varphi(x,y)\,S,\\
&L\to\varphi(x,y)\,L, &&X\to\varphi(x,y)\,X,\\
&Y\to\varphi(x,y)\,Y
\end{aligned} \tag{4.4}$$

since numerator and denominator of the fraction in (1.5) can be multiplied by the same factor $\varphi(x,y)$. Substituting (4.4) into the formulas (3.8) and (3.9) we get gauge transformations for M and N:

$$M\to\varphi(x,y)^4\,M, \qquad\qquad N\to\varphi(x,y)^4\,N. \tag{4.5}$$

Now let's substitute (4.4) and (4.5) into (3.13). This gives gauge transformations for the field Ω:

$$\Omega\to\varphi(x,y)^5\,\Omega. \tag{4.6}$$

From (4.4), (4.5), and (4.6), on substituting them into (4.1), we find that the function $k(z)$ is invariant under gauge transformations (4.4). Hence polynomial (4.2) is also invariant under such transformations. Theorem 4.1 is proved. □

Now we shall derive the transformation rules for P^*, Q^*, R^*, and S^* under the point transformations (1.2). Let's substitute $z = y'$ into the fraction (4.1) and let's apply the formula (2.2) for the derivative y'. Taking into account the transformation rules (2.5), (2.10), and (3.10) we get

$$k(y') = \frac{\det S}{(x_{1.0} + x_{0.1}\,\tilde{y}')^3}\,\tilde{k}(\tilde{y}'). \tag{4.7}$$

Here $\tilde{k}(z)$ is a fraction of the form (4.1) with X, Y, M, N, and Ω replaced by $\tilde{X}$, $\tilde{Y}$, $\tilde{M}$, $\tilde{N}$, and $\tilde{\Omega}$. In order to get transformation rules for P^*, Q^*, R^*, S^* let's write the equation (1.5) as follows:

$$y'' = P^*(x,y) + 3\,Q^*(x,y)\,y' + 3\,R^*(x,y)\,{y'}^2 + S^*(x,y)\,{y'}^3 + k(y'). \tag{4.8}$$

Now we substitute (2.2) and (2.3) into (4.8) and take into account (4.7). This leads us to the following central result of our paper.

Theorem 4.2. *Coefficients $P^*(x,y)$, $Q^*(x,y)$, $R^*(x,y)$, and $S^*(x,y)$ in polynomial* (4.2) *are transformed exactly as the coefficients of the differential equation*

$$y'' = P^*(x,y) + 3\,Q^*(x,y)\,y' + 3\,R^*(x,y)\,{y'}^2 + S^*(x,y)\,{y'}^3 \tag{4.9}$$

under the point transformations of the form (1.2).

This equation (4.9) is called *canonically associated* equation for (1.5). It is of the form (1.1). From theorems 4.1 and 4.2 we get the following obvious result.

Theorem 4.3. *If two equations of the form* (1.5) *are point-equivalent, i.e. one of them is obtained from another by some point transformation* (1.2)*, then their associated equations* (4.9) *are also point-equivalent.*

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