MAGNETIZATION WAVES IN THE LANDAU-LIFSHITZ MODEL

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Abstract. The solutions of the Landau-Lifshitz equation with finite-gap behavior at infinity are considered. By means of the inverse scattering method the large-time asymptotics is obtained.

1. The Landau-Lifshitz equation [1] describing the dynamics of the magnetization vector $\mathbf{S}$ for the one-dimensional ferromagnet of the "light-plane" type can be written in the following form:

$$\mathbf{S}_t=\mathbf{S}\times\mathbf{S}_{xx}+\mathbf{S}\times J\mathbf{S},\quad |\mathbf{S}|=1,\quad J=\operatorname{diag}(0,0,-16\omega^2). \tag{1}$$

In [2, 3] equation (1) was shown to be completely integrable and it was represented as a compatibility condition for the pair of linear equations

$$\partial_x\Psi=U\Psi,\qquad\qquad \partial_t\Psi=V\Psi \tag{2}$$

with 2×2 matrices U and V of the form

$$\begin{aligned}U&=-i\sum_{\alpha=1}^{3}S^\alpha\,w_\alpha\,\sigma_\alpha,\\ V&=2i\sum_{\alpha=1}^{3}\frac{w_1\,w_2\,w_3}{w_\alpha}\,S^\alpha\,\sigma_\alpha-i\sum_{\alpha=1}^{3}[\mathbf{S}\times\mathbf{S}_x]^\alpha\,w_\alpha\,\sigma_\alpha,\end{aligned}$$

where σ_α are the Pauli matrices and $w_1=w_2=\sqrt{\lambda^2-\omega^2}$, $w_3=\lambda$. Soliton-like solutions of (1) are well-known (see [4, 5]). The class of periodic and almost periodic wave-like solutions of (1) contains an important subclass of algebro-geometric (or finite-gap) solutions. They were constructed in [5, 6]. The study of algebro-geometric solutions for integrable equations was initiated by Novikov in [7], it led to the well-developed theory of finite-gap integration (see review [8]).

In this paper we study the large-time asymptotics for "nearly finite-gap solutions" of the Landau-Lifshitz equation, i. e. the solutions $\mathbf{S}$ with the following behavior as $x\to\pm\infty$:

$$\begin{aligned}\mathbf{S}(x,t)&\to S(x,t\,|\,\Gamma,D_1,\delta_1),\quad x\to+\infty,\\ \mathbf{S}(x,t)&\to S(x,t\,|\,\Gamma,D_2,\delta_2),\quad x\to-\infty.\end{aligned} \tag{3}$$

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Here $S(x,t\,|\,\Gamma,\delta)$ denotes a real smooth g-gap solution of (1) with a phase δ constructed on a base of the hyperelliptic Riemann surface Γ with a fixed divisor $D = P_1 + \ldots + P_g$ on it. Given the branching points

$$\lambda_0 = -\omega < \lambda_1 < \lambda_2 < \ldots < \lambda_{2g} < \omega = \lambda_{2g+1}$$

of Γ one can define the meromorphic function

$$Y = \sqrt{(\lambda^2 - \omega^2)(\lambda - \lambda_1)\cdot\ldots\cdot(\lambda - \lambda_{2g})}$$

on Γ and the pair of infinity points $P^{\pm}_{\infty}$, with $Y \sim \pm\lambda^{g+1}$ as $P \to P^{\pm}_{\infty}$. The Riemann surface Γ consists of two sheets: Γ_+ (upper sheet) and Γ_- (lower sheet). It admits of the hyperelliptic involution σ, which does interchange sheets, and the antiholomorphic involution τ, $(\lambda(\tau P) = \overline{\lambda(P)},\quad Y(\tau P) = -\overline{Y(P)})$, which does not. The boundary $\partial\Gamma_+$ is a collection of g cycles $\gamma_1, \ldots, \gamma_g$ and the cycle γ_∞ passing through two infinity points $P^{\pm}_{\infty}$.

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Fig. 1.

Let us choose the canonical basis of cycles a_i, b_i, $i = l, \ldots, g$ on Γ as it is shown on fig. 1. The finite-gap solution $S(x,t\,|\,\Gamma,\delta)$ then is given up to a phase shift by explicit formulae in terms of Riemann θ-functions:

$$S^1 = \frac{C_1C_2 - C_3C_4}{C_3C_2 - C_1C_4}, \qquad S^2 = -i\frac{C_1C_2 + C_3C_4}{C_3C_2 - C_1C_4}, \qquad S^3 = \frac{C_3C_2 + C_4C_1}{C_3C_2 - C_1C_4},$$

Here

$$\begin{aligned}
&C_1 = \theta[n,0](\Omega + \Delta + z), && C_3 = \theta(\Omega + \Delta + z),\\
&C_2 = -\theta[n,0](\Omega + \Delta - z), && C_4 = \theta(\Omega + \Delta - z),\\
&n = \frac{1}{2}(1,0,\ldots,0).
\end{aligned}$$

The change of phase δ is equivalent to the rotation of the vector $\mathbf{S}$ around the third coordinate axis. The vector Δ is connected with the divisor by the Abel map

$$A\colon \mathrm{div}(\Gamma) \to \mathrm{Jac}(\Gamma), \quad A_i(P) = \int_{\lambda_0}^{P} \omega_i, \quad P \in \Gamma,$$

according to the formula $\Delta = -A(D) - K$, where K is the vector of Riemann constants. Real solutions $\mathbf{S}(x,t)$ corresponding to real divisors are determined by the restrictions

$$A(D) - A(\tau D) = A(\lambda_0 + \lambda_{2g+1} - P_\infty^+ - P_\infty^-) = 0. \tag{4}$$

Vector $Q = i(V^{(1)}x - V^{(2)}t)$ is composed of two vectors $V^{(1)}$ and $V^{(2)}$, being the vectors b-periods of two normalized abelian differentials of the second kind with the only poles at infinities $P_\infty^\pm$. These differentials have the following leading terms of Laurent expansions at these points:

$$\Omega^{(1)} = \mp d\lambda + \dots, \qquad \Omega^{(2)} = \pm 4\,\lambda\, d\lambda + \dots.$$

Vector $z \in \mathrm{Jac}(\Gamma)$ is equal to $A(P_\infty^+)$, the path of integration γ is shown on fig. 1.

The reality condition (4) defines 2^g disjoint real tori T_ν, $\nu = 0, \dots, 2^g - 1$ in $\mathrm{Jac}(\Gamma)$. We choose only one of them: torus T_0 with

$$\mathrm{Re}[\Delta + A(\lambda_0)] = 0,$$

on which the θ-function $\theta(A(\lambda_0) + \Omega + \Delta)$ does not vanish (see [9]). The main instrument in constructing finite-gap solutions is the matrix Baker-Akhiezer function

$$e(P) = \begin{Vmatrix} e_1^+(P) & e_1^+(\sigma P) \\ e_2^+(P) & e_2^+(\sigma P) \end{Vmatrix}$$

solving equations (2). The first column of it is given up to a scalar multiples $f_1(x,t)$ and $f_2(x,t)$ by formulas

$$\begin{aligned} e_1^+(P) &= f_1\, e^{i\delta/2}\, \frac{\theta(A(\lambda) + \Omega + \Delta)}{\theta(A(\lambda) + \Delta)}\, \exp\left(i \int_{\lambda_0}^{P} (\Omega^{(1)}x + \Omega^{(2)}t) \right), \\ e_1^+(P) &= f_2\, e^{-i\delta/2}\, \frac{\theta[n,0](A(\lambda) + \Omega + \Delta)}{\theta(A(\lambda) + \Delta)}\, \exp\left(i \int_{\lambda_0}^{P} (\Omega^{(1)}x + \Omega^{(2)}t) \right). \end{aligned} \tag{6}$$

Multiples $f_1(x,t)$ and $f_2(x,t)$ are defined by fixing $\det e(P)$ and by the condition $e_1(\lambda_0)/e_2(\lambda_{2g+1}) = e^{i\delta}$.

Remark. The torus T_0 is an exceptional real torus in the following sense: Baker-Akhiezer function $e(P,x,t)$ is non-singular bounded function in x,t for $P \in \partial\Gamma_+$.

2. In order to construct a scattering theory for $\mathbf{S}(x,t)$ of the form (3) let us define the vectorial Jost functions $\varPhi(P)$ and $\varPsi(P)$ solving (2) and having asymptotics

$$\begin{aligned} \varPhi(P) &\to e^{+}(P, D_1, \delta_1) \quad \text{as} \quad x \to +\infty,\\ \varPsi(P) &\to e^{+}(P, D_2, \delta_2) \quad \text{as} \quad x \to -\infty. \end{aligned}$$

The functions $\varPhi$, $\varPsi$ are bounded with each other by scattering data $a(P)$, $b(P)$:

$$\varPhi(P) = \varPsi(P)\, a(P) + \varPsi(\sigma P) b(P), \quad P \in \partial\Gamma_{+}. \tag{7}$$

In this paper we study the non-soliton case, i. e. $a(P) \neq 0$, if $P \in \Gamma_{+}$. Starting from (7) we obtain a scattering theory for (1), (3) most similar to that of [10] for the fast-decreasing case. The only difference consists in the existence of relations between asymptotic divisors D_1, D_2, phases δ_1, δ_2 and scattering data $a(P)$, $b(P)$:

$$\begin{aligned} &A(D_2 - D_1) = \frac{1}{2\pi i} \int\limits_{\partial\Gamma_{+}} \ln|1 - r(P)\, r(\sigma P)|\, \omega(P),\\ &\delta_1 - \delta_2 = -i \ln\left(\frac{a(\lambda_{2g+1}) + b(\lambda_{2g+1})}{a(\lambda_0) - b(\lambda_0)}\right). \end{aligned} \tag{8}$$

Here $r(P) = b(P)/a(P)$ is the reflection coefficient. It should be pointed out that for our choice of divisors D_1 and D_2 (i. e. torus T_0) $1 - r(P)\, r(\sigma P)$ is a real and positive function on $\partial\Gamma_{+}$.

For the asymptotical analysis of (1), (3) we use a singular integral equation for Jost functions similar to that of [11]. Our method is a generalization of the asymptotical construction of [12].

The final result of our investigation is the following: the main term of the asymptotics for $S(x,t)$ as $t \to +\infty$ is given by the finite-gap solution

$$\mathbf{S}(x,t) = \mathbf{S}(x, t\,|\, D(\xi), \delta(\xi)) + \varepsilon(\xi, t), \quad \varepsilon(\xi, t) = o(1),$$

with the phase $\delta(\xi)$ and divisor $D(\xi)$ depending on the "slow variable" $\xi = x/t$ according to

$$\begin{aligned} &A(D(\xi)) = A(D_2) - \frac{1}{2\pi i} \int\limits_{\ell(\xi)} \ln|1 - r(P)\, r(\sigma P)|\, \omega(P),\\ &\delta(\xi) = \delta_2 - i \ln\left(\frac{\tilde{A}(\lambda_{2g+1})}{\tilde{A}(\lambda_0)}\, \frac{1 + \tilde{r}(\lambda_{2g+1})}{1 - \tilde{r}(\lambda_0)}\right). \end{aligned} \tag{10}$$

Here the path of integration $\ell(\xi)$ is a part of the contour $\partial\Gamma_{+}$ which is situated to the left of the unique stationary point $P_0(\xi)$ (see fig. 1) defined by the condition

$$\left(\Omega^{(1)}\xi + \Omega^{(2)}\right)\Big|_{P=P_0} = 0.$$

The function $\tilde{A}(P)$ is given by

$$\tilde{A}(P) = \lim_{P' \to P} \alpha(P'), \quad P' \in \Gamma_+, \quad P \in \partial\Gamma_+,$$

$$\alpha(P) = \frac{\theta((A(P) - A(D(\xi)) - K)}{\theta((A(P) - A(D_2) - K)} \exp\left(-\frac{1}{2\pi i} \int\limits_{\ell(\xi)} M(Q,P) \ln(1 - r(Q)\, r(\sigma Q)) \right),$$

where $M(Q,P)$ is the multivalued Cauchy kernel (see [9]). The function $\tilde{r}(P)$ is given by

$$\tilde{r}(P) = \begin{cases} r(P) & \text{for} \quad P \in \ell(\xi), \\ 0 & \text{for} \quad P \notin \ell(\xi). \end{cases}$$

The value of the rest term of asymptotics (9) depends on whether $P_0 \in \partial\Gamma_+$ or not. In the first case $\varepsilon = O(t^{-1/2})$, in the second case $\varepsilon = o(t^{-N})$ for any $N > 0$. The scattering problem studied here describes the interaction of two magnetization waves with the same spectrum Γ. After finishing all the "transition processes" two interacting waves consolidate into one asymptotical wave (9) with slowly changing phases.

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