HAMILTONIAN APPROACH TO DERIVING THE GRAVITY EQUATIONS FOR A 3D-BRANE UNIVERSE.

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Abstract. Recently, using the concept of temporal coexistence, some arguments were suggested saying that our universe should be considered as a three-dimensional brane equipped with a Riemannian metric depending on the cosmological time. The Lagrangian approach to this 3D-brane model of the universe shows that the number of gravity equations in this model is less than it follows from Einstein's equation written in 3D+1 presentation thus making this 3D-brane model a separate non-Einsteinian theory of gravitation. In the present paper we continue the research of this theory developing a Hamiltonian approach to it.

1. Introduction.

In the 3D-brane paradigm suggested and argued in [1] (see also [2] and [3]) the gravitational field is described by a time-dependent 3D metric with the components

$$g_{ij} = g_{ij}(x^0, x^1, x^2, x^3), \quad 1 \leqslant i, j \leqslant 3, \tag{1.1}$$

where $x^0 = c\,t$ and c is the speed of light. This 3D paradigm is related to the standard 4D paradigm through the metric

$$G_{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -g_{11} & -g_{12} & -g_{13} \\ 0 & -g_{21} & -g_{22} & -g_{23} \\ 0 & -g_{31} & -g_{32} & -g_{33} \end{Vmatrix}, \qquad G^{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -g^{11} & -g^{12} & -g^{13} \\ 0 & -g^{21} & -g^{22} & -g^{23} \\ 0 & -g^{31} & -g^{32} & -g^{33} \end{Vmatrix}. \tag{1.2}$$

In the standard paradigm t is interpreted as the cosmological time (see [4]), while x^1, x^2, x^3 are interpreted as comoving coordinates (see [5]).

In the standard four-dimensional paradigm of general relativity and cosmology the four-dimensional metric should obey the standard Einstein's equation

$$r_{ij} - \frac{r}{2}\,G_{ij} - \Lambda\,G_{ij} = \frac{8\,\pi\,\gamma}{c^4}\,T_{ij}, \tag{1.3}$$

(see § 2 in Chapter V of [6]). Here c is the speed of light, γ is Newton's gravitational constant (see [7]), and Λ is the cosmological constant (see [8]). The quantities T_{ij} in the right hand side of (1.3) are the components of the energy-momentum tensor (see

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[9]). The term r_{ij} in (1.3) corresponds to the components of the four-dimensional Ricci tensor and r is the four-dimensional scalar curvature (see § 8 in Chapter IV of [10]). By substituting (1.2) into (1.3) in [1] the following equations were derived:

$$\begin{aligned}&\frac{\partial b_{ij}}{\partial x^0}-\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\\&-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{8\,\pi\,\gamma}{c^4}\,T_{ij},\end{aligned}\tag{1.4}$$

$$\sum_{k=1}^{3}\nabla_k\,b^k_j-\sum_{k=1}^{3}\nabla_j\,b^k_k=\frac{8\,\pi\,\gamma}{c^4}\,T_{0j},\tag{1.5}$$

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\frac{R}{2}-\Lambda=\frac{8\,\pi\,\gamma}{c^4}\,T_{00}.\tag{1.6}$$

Here R_{ij} are the components of the three-dimensional Ricci tensor, R is the three-dimensional scalar curvature, and b_{ij} are given by the formula

$$b_{ij}=\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}=\frac{\dot g_{ij}}{2\,c}.\tag{1.7}$$

Later on in [11] the Lagrangian approach to the 3D-brane model was applied. For this purpose the standard 4D action integral was taken

$$S_{\text{gr}}=-\frac{c^3}{16\pi\gamma}\int(r+2\,\Lambda)\,\sqrt{-\det G}\;d^4x\tag{1.8}$$

(see § 2 in Chapter V of [6]) and then it was transformed to the 3D form

$$S_{\text{gr}}=-\frac{c^3}{16\pi\gamma}\iint(r+2\,\Lambda)\,\sqrt{\det g}\;d^3x\,dx^0.\tag{1.9}$$

The scalar curvature r in (1.8) and (1.9) is associated with the four-dimensional metric (1.2). As is was shown in [1], it is expressed through the three-dimensional scalar curvature R in the following way:

$$r=-2\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}-R-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,.\tag{1.10}$$

The action integral (1.9) with r given by the formula (1.10) was complemented with the action integral responsible for matter:

$$S_{\text{mat}}=\iint\mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x\,dx^0.\tag{1.11}$$

Taking the total action in the form of the sum

$$S=S_{\text{gr}}+S_{\text{mat}}\tag{1.12}$$

and applying the stationary-action principle (see [12]) to (1.12), in [11] the equation (1.4) was rederived along with the following purely three-dimensional expression for the components of the energy-momentum tensor in it:

$$T_{ij} = -2\, c\, \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{ij}}. \tag{1.13}$$

As for the equations (1.5) and (1.6), they were omitted since they cannot be derived within the purely three-dimensional Lagrangian approach to the theory.

The main goal of the present paper is to develop a Hamiltonian approach to deriving the equation (1.4).

2. Reducing the order of the action integral.

Most physical theories lead to differential equations of the second order with respect to time derivatives. Their action integrals (if any) are of the first order in time derivatives. However, looking at (1.9), we see that S_{gr} is different due to the first term in the right hand side of (1.10). Indeed, due to (1.7) we have

$$-2\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0} = -\sum_{k=1}^{3}\frac{\partial}{\partial x^0}\biggl(\,\sum_{q=1}^{3}\frac{\partial g_{kq}}{\partial x^0}\, g^{kq}\biggr), \tag{2.1}$$

where $x^0 = c\,t$. In order to reduce the order of time derivatives in (2.1) we apply integration by parts in the action integral (1.9):

$$\int\limits_v^u\biggl(\int\frac{\partial b^k_k}{\partial x^0}\,\sqrt{\det g}\;d^3x\biggr)\,dx^0 = \biggl|_v^u\int b^k_k\,\sqrt{\det g}\;d^3x - \int\limits_v^u\biggl(\int b^k_k\,\frac{\partial(\sqrt{\det g}\,)}{\partial x^0}\;d^3x\biggr)\,dx^0.$$

Non-integral terms usually do not affect differential equations derived from action integrals. Therefore we can replace the action integral (1.9) with the following one:

$$S_{\text{gr}} = -\frac{c^3}{16\pi\gamma}\iint(\,\rho + 2\,\Lambda)\,\sqrt{\det g}\;d^3x\,dx^0, \tag{2.2}$$

where

$$\rho = 2\sum_{k=1}^{3}\frac{b^k_k}{\sqrt{\det g}}\,\frac{\partial(\sqrt{\det g}\,)}{\partial x^0} - R - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q\,. \tag{2.3}$$

In order to transform (2.3) we use the formula

$$\frac{\partial(\sqrt{\det g}\,)}{\partial x^0} = \frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\frac{\partial g_{kq}}{\partial x^0}\,\sqrt{\det g}\,. \tag{2.4}$$

This formula (2.4) is easily derived from the well-known Jacobi's formula for differentiating determinants (see [13]). Applying (1.7) to (2.4), we get

$$\frac{\partial(\sqrt{\det g}\,)}{\partial x^0} = \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\, b_{kq}\,\sqrt{\det g} = \sum_{q=1}^{3} b^q_q\,\sqrt{\det g}\,. \tag{2.5}$$

Substituting (2.5) into (2.3), we find

$$\rho = \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q - R - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k. \tag{2.6}$$

By means of direct calculations it is easy to show that the action integral (2.2) with the function ρ given by the formula (2.6) leads to the same differential equation (1.4) as the action integral (1.9) with the function r given by the formula (1.10). But unlike (1.10), the formula (2.6) has no derivatives of the tensor field $\mathbf{b}$.

3. Legendre transformation.

In classical mechanics the Legendre transformation is used for converting Lagrangian mechanics into Hamiltonian mechanics (see [14]). Below we develop field-theoretic version of this transformation and apply it to the 3D-brane model of gravity. Relying on (2.2), (2.6), (1.11), and (1.12), we define

$$L=-\frac{c^3}{16\pi\gamma}\int(\,\rho+2\,\Lambda)\,\sqrt{\det g}\;d^3x+\int\mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x. \tag{3.1}$$

Upon substituting (2.6) into (3.1) we transform (3.1) as follows:

$$\begin{aligned}L=\frac{c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}b_{ij}\,g^{ik}\,b_{kq}\,g^{jq}\,\sqrt{\det g}\;d^3x\,-\\-\frac{c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}b_{ij}\,g^{ij}\,b_{kq}\,g^{kq}\,\sqrt{\det g}\;d^3x\,-\\-\frac{c^3}{16\pi\gamma}\int(2\,\Lambda-R)\,\sqrt{\det g}\;d^3x+\int\mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{3.2}$$

The quantities b_{ij} are related to g_{ij} through the formula (1.7). However in this section we treat them as independent dynamic variables. The same trick is used in Lagrangian mechanics (see [15]) where generalized coordinates and their time derivatives are considered as independent arguments of the Lagrange function

$$L=L(q_1,\,\dots\,,q_n,\dot{q}_1,\,\dots\,,\dot{q}_n). \tag{3.3}$$

The function (3.2) below plays the same role as the Lagrange function (3.3) in Lagrangian mechanics.

The function $\mathcal{L}_{\text{mat}}$ in (3.2) describes matter. It is different for different sorts of matter. Typically it does not depend on b_{ij}. But it can depend on g_{ij} and on spatial derivatives of g_{ij}. Apart from g_{ij} the function $\mathcal{L}_{\text{mat}}$ in (3.2) depends on dynamical variables describing matter and on their time derivatives. We denote them through $Q_1,\,\dots\,,\,Q_n$ and introduce the following notations for their time derivatives:

$$W_i=\frac{\partial Q_i}{\partial x^0}=\frac{\dot{Q}_i}{c},\qquad i=1,\,\dots\,,\,n. \tag{3.4}$$

The index i in (3.4) just enumerates the dynamical variables of matter. It does not take into account their transformational behavior. They can be components of some tensorial and/or spinor fields, they can be components of some sections in complex vector bundles associated with electromagnetic, weak, and strong interactions (see [16]). In the case of dark matter their structure is yet unknown.

The formula (3.4) is similar to the formula (1.7). Using it, we write

$$\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}}(Q_1, \ldots, Q_n, W_1, \ldots, W_n, \mathbf{g}). \tag{3.5}$$

Writing Q_i in the argument list of the function $\mathcal{L}_{\text{mat}}$ in (3.5) we assume that $\mathcal{L}_{\text{mat}}$ depends not only on Q_i, but on some finite number of spacial derivatives[1] of the function $Q_i(x^0, x^1, x^2, x^3)$. The same assumption applies to each argument $W_i = W_i(x^0, x^1, x^2, x^3)$ and to each component (1.1) of the metric $\mathbf{g}$ in (3.5).

Remark. Generally speaking the function (3.5) can depend on the components of the tensor field $\mathbf{b}$ too. However in most cases it does not.

Apart from $\mathcal{L}_{\text{mat}}$, in (3.2) we have the function $\mathcal{L}_{\text{gr}}(\mathbf{g}, \mathbf{b})$ responsible for gravity. This function is given explicitly through the formula

$$\mathcal{L}_{\text{gr}} = \frac{c^3}{16\pi\gamma}\biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(b_{ij}\, g^{ik}\, b_{kq}\, g^{jq} - b_{ij}\, g^{ij}\, b_{kq}\, g^{kq}\bigr) + R - 2\,\Lambda\biggr). \tag{3.6}$$

In terms of (3.5) and (3.6) the formula (3.2) is written as

$$L = \int \mathcal{L}\,\sqrt{\det g}\; d^3x, \ \text{ where } \ \mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}. \tag{3.7}$$

The arguments of the function $\mathcal{L}$ in (3.7) are written as follows:

$$\mathcal{L} = \mathcal{L}(Q_1, \ldots, Q_n, W_1, \ldots, W_n, \mathbf{g}, \mathbf{b}). \tag{3.8}$$

We need to define partial variational derivatives for functions like (3.5), (3.6), and (3.8). Let's introduce small variations to the arguments $W_1, \ldots, W_n$ of them:

$$\hat{W}_i = W_i(x^0, x^1, x^2, x^3) + \varepsilon\, h_i(x^0, x^1, x^2, x^3) \tag{3.9}$$

Here $\varepsilon \to 0$ is a small parameter, while $h_i(x^0, x^1, x^2, x^3)$ are smooth functions with compact support (see [17]). Substituting (3.9) into the arguments of (3.8) and then substituting (3.8) into the integral (3.7), we get

$$\hat{L} = L + \varepsilon \int \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} h_i\,\sqrt{\det g}\; d^3x + \ldots. \tag{3.10}$$

Here in (3.10) and in what follows below through dots we denote higher order terms with respect to the small parameter ε. Similarly, we can introduce small variations to the arguments $Q_1, \ldots, Q_n$ in (3.8):

$$\hat{Q}_i = Q_i(x^0, x^1, x^2, x^3) + \varepsilon\, h_i(x^0, x^1, x^2, x^3). \tag{3.11}$$

Despite the relationships (3.4) the functions $W_1, \ldots, W_n$ and $Q_1, \ldots, Q_n$ in (3.9) and (3.11) are treated as independent functions. Substituting (3.11) into the arguments of (3.8) and then substituting (3.8) into the integral (3.7), we get

$$\hat{L} = L + \varepsilon \int \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}} h_i\,\sqrt{\det g}\; d^3x + \ldots. \tag{3.12}$$

[1] Spacial derivatives are derivatives with respect to the spacial coordinates x^1, x^2, x^3. The variable $x^0 = c\,t$ is associated with the time variable t.

Small variations of the metric $\mathbf{g}$ are introduced through the formulas

$$\hat{g}_{ij} = g_{ij}(x^0, x^1, x^2, x^3) + \varepsilon\, h_{ij}(x^0, x^1, x^2, x^3), \tag{3.13}$$

Substituting (3.13) for the components of the metric $\mathbf{g}$ into (3.8) and then substituting (3.8) into the integral (3.7), we can write

$$\hat{L} = L + \varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Big(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Big)_{\mathbf{Q},\mathbf{W},\mathbf{b}} h_{ij}\,\sqrt{\det g}\; d^3x + \ldots . \tag{3.14}$$

And finally we introduce small variations to the components of the tensor field $\mathbf{b}$:

$$\hat{b}_{ij} = b_{ij}(x^0, x^1, x^2, x^3) + \varepsilon\, h_{ij}(x^0, x^1, x^2, x^3), \tag{3.15}$$

Substituting (3.15) for the components of the field $\mathbf{b}$ into (3.8) and then substituting (3.8) into the integral (3.7), we can write

$$\hat{L} = L + \varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Big(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Big)_{\mathbf{Q},\mathbf{W},\mathbf{g}} h_{ij}\,\sqrt{\det g}\; d^3x + \ldots . \tag{3.16}$$

Thus we have introduced partial variational derivatives

$$\Big(\frac{\delta\mathcal{L}}{\delta W_i}\Big)_{\mathbf{Q},\mathbf{g},\mathbf{b}},\ \Big(\frac{\delta\mathcal{L}}{\delta Q_i}\Big)_{\mathbf{W},\mathbf{g},\mathbf{b}},\ \Big(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Big)_{\mathbf{W},\mathbf{Q},\mathbf{b}},\ \Big(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Big)_{\mathbf{W},\mathbf{Q},\mathbf{g}}. \tag{3.17}$$

The relationships (3.10), (3.12), (3.14), and (3.16) serve as definitions of the derivatives (3.17). The corresponding derivatives for the functions (3.5) and (3.6)

$$\Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta W_i}\Big)_{\mathbf{Q},\mathbf{g},\mathbf{b}},\ \Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta Q_i}\Big)_{\mathbf{W},\mathbf{g},\mathbf{b}},\ \Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Big)_{\mathbf{Q},\mathbf{W},\mathbf{b}},\ \Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Big)_{\mathbf{Q},\mathbf{W},\mathbf{g}}, \tag{3.18}$$

$$\Big(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta W_i}\Big)_{\mathbf{Q},\mathbf{g},\mathbf{b}},\quad \Big(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta Q_i}\Big)_{\mathbf{W},\mathbf{g},\mathbf{b}},\quad \Big(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Big)_{\mathbf{Q},\mathbf{W},\mathbf{b}},\quad \Big(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Big)_{\mathbf{Q},\mathbf{W},\mathbf{g}}. \tag{3.19}$$

are defined similarly (compare (3.18) and (3.19) with (3.17).

Generally speaking the variational derivative in (1.13) is different from those defined in (3.18). However it can be expressed through them:

$$\begin{aligned}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}} = &- \frac{1}{2}\,\frac{\partial}{\partial x^0}\Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Big)_{\mathbf{W},\mathbf{Q},\mathbf{g}} - \\ &- \frac{1}{2}\,\Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Big)_{\mathbf{W},\mathbf{Q},\mathbf{g}} \sum_{q=1}^{3} b^q_q + \Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Big)_{\mathbf{W},\mathbf{Q},\mathbf{b}},\end{aligned} \tag{3.20}$$

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}} = -\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{kq}}\, g_{ik}\, g_{qj}. \tag{3.21}$$

If we recall the remark on page 5 and look at (3.5), we see that $\mathcal{L}_{\text{mat}}$ does not depend on b_{ij}. In this special case

$$\Big(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Big)_{\mathbf{W},\mathbf{Q},\mathbf{g}} = 0. \tag{3.22}$$

Applying (3.22) to (3.20), in this special case we reduce (3.20) to

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}=\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}. \tag{3.23}$$

In classical mechanics the Legendre transformation consists in replacing generalized velocities $\dot q_1,\,\ldots\,,\,\dot q_n$ by the generalized momenta:

$$p_i=\frac{\partial L}{\partial \dot q_i},\quad i=1,\,\ldots\,,\,n. \tag{3.24}$$

Here L is the Lagrange function from (3.3). In our present case the Lagrange function is given by the formula (3.8). By analogy to (3.24) here we define the Legendre transformation through the formulas

$$\beta^{ij}=\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}, \qquad\qquad P^i=\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}. \tag{3.25}$$

The momenta β^{ij} are used in order to replace the quantities b_{ij} and the momenta P^i are used in order to replace the quantities W_i.

4. The energy function and the Hamiltonian.

In classical mechanics the energy function is defined through the formula

$$H=\sum_{i=1}^{n}p_i\,\dot q_i-L, \tag{4.1}$$

where L is the Lagrange function (3.3) and $p^1,\,\ldots\,,\,p^n$ are the generalized momenta given by the formula (3.24).

Definition 4.1. The Hamiltonian or the Hamilton function in classical mechanics is the energy function (4.1) expressed through the variables $q_1,\,\ldots\,,\,q_n,\,p^1,\,\ldots\,,\,p^n$.

By analogy to (4.1) we define the energy function through the formula

$$H=\int\Biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\sum_{i=1}^{n}P^i\,W_i\Biggr)\sqrt{\det g}\;d^3x-L. \tag{4.2}$$

Here L is given by the formula (3.7), while β^{ij} and P^i are defined through the formulas (3.25). We write the formula (4.2) as

$$H=\int\mathcal{H}\,\sqrt{\det g}\;d^3x, \tag{4.3}$$

where

$$\mathcal{H}=\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\sum_{i=1}^{n}P^i\,W_i-\mathcal{L}. \tag{4.4}$$

Assuming that the Legendre transformation (3.25) is invertible, we consider the function (4.4) as a function with the following arguments

$$\mathcal{H}=\mathcal{H}(Q_1,\ldots,Q_n,P^1,\ldots,P^n,\mathbf{g},\boldsymbol{\beta}). \tag{4.5}$$

Each argument Q_i in the argument list of the function (4.5) represents the function

$Q_i(x^0, x^1, x^2, x^3)$ and some finite number of its spacial[1] derivatives. The same is true for each argument $P^i = P^i(x^0, x^1, x^2, x^3)$ in (4.5), for each component of the metric $\mathbf{g}$, and for each component of the tensor field $\boldsymbol{\beta}$ in (4.5).

Definition 4.2. The function (4.4) written in the form of (4.5) is called the Hamilton function or the Hamiltonian of gravity and matter in the 3D-brane model.

For the function (4.5) the following partial variational derivatives are defined:

$$\Bigl(\frac{\delta\mathcal{H}}{\delta P^i}\Bigr)_{\mathbf{Q},\mathbf{g},\boldsymbol{\beta}},\ \Bigl(\frac{\delta\mathcal{H}}{\delta Q_i}\Bigr)_{\mathbf{P},\mathbf{g},\boldsymbol{\beta}},\ \Bigl(\frac{\delta\mathcal{H}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\boldsymbol{\beta}},\ \Bigl(\frac{\delta\mathcal{H}}{\delta \beta^{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}}. \tag{4.6}$$

The derivatives (4.6) are introduced through formulas similar to the formulas (3.10), (3.12), (3.14), and (3.16).

Theorem 4.1. *If the Legendre transformation* (3.25) *is invertible, then the inverse transformation is given by the formulas*

$$b_{ij} = \Bigl(\frac{\delta\mathcal{H}}{\delta \beta^{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}}, \qquad\qquad W_i = \Bigl(\frac{\delta\mathcal{H}}{\delta P^i}\Bigr)_{\mathbf{Q},\mathbf{g},\boldsymbol{\beta}}. \tag{4.7}$$

Proof. Keeping P_i, Q_i, and g_{ij} unchanged, we introduce small variations to β^{ij}:

$$\hat\beta^{ij} = \beta^{ij}(x^0, x^1, x^2, x^3) + \varepsilon\, h^{ij}(x^0, x^1, x^2, x^3). \tag{4.8}$$

Invertibility of the Legendre transformation (3.25) means that the variations (4.8) induce small variations of b_{ij} and small variations of the variables $W_1, \ldots, W_n$:

$$\hat b_{ij} = b_{ij}(x^0, x^1, x^2, x^3) + \varepsilon\, \tilde h_{ij}(x^0, x^1, x^2, x^3), \tag{4.9}$$

$$\hat W_i = W_i(x^0, x^1, x^2, x^3) + \varepsilon\, \tilde h_i(x^0, x^1, x^2, x^3). \tag{4.10}$$

The functions $\tilde h_{ij}(x^0, x^1, x^2, x^3)$ and $\tilde h_i(x^0, x^1, x^2, x^3)$ in (4.9) and (4.10) are determined by the functions $h^{ij}(x^0, x^1, x^2, x^3)$ in (4.8). Applying (4.8) to the integral (4.3) with the function $\mathcal{H}$ written as (4.5), we get

$$\hat H = H + \varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{H}}{\delta \beta^{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}} h^{ij}\,\sqrt{\det g}\;d^3x + \ldots\,. \tag{4.11}$$

Applying (4.8), (4.9), and (4.10) to the same integral (4.3) with the function $\mathcal{H}$ written as (4.4) and taking into account (3.7) and (3.8), we derive

$$\begin{gathered}\hat H = H + \varepsilon \int \Biggl(\,\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(h^{ij}\, b_{ij} + \beta^{ij}\,\tilde h_{ij}\Bigr) + \sum_{i=1}^{n} P^i\,\tilde h_i\Biggr)\sqrt{\det g}\;d^3x\,-\\ -\,\varepsilon \int \Biggl(\,\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\tilde h_{ij} + \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\tilde h_i\Biggr)\sqrt{\det g}\;d^3x + \ldots\,.\end{gathered}\tag{4.12}$$

[1] Spacial derivatives are derivatives with respect to the spacial coordinates x^1, x^2, x^3. The variable $x^0 = c\,t$ is associated with the time variable t.

If we take into account the relationships (3.25), then the formula (4.12) reduces to

$$\hat{H}=H+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}b_{ij}\,h^{ij}\,\sqrt{\det g}\,d^3x+\ldots. \tag{4.13}$$

Now, comparing (4.13) with (4.11) as $\varepsilon\to 0$, we find that the first of the two formulas (4.7) is proved.

In order to prove the second formula (4.7) we keep β^{ij}, g_{ij}, and Q_i unchanged and introduce small variations to the variables $P^1,\,\ldots\,,\,P^n$:

$$\hat{P}^i=P^i(x^0,x^1,x^2,x^3)+\varepsilon\,h^i(x^0,x^1,x^2,x^3). \tag{4.14}$$

Invertibility of the Legendre transformation (3.25) again means that the variations (4.14) induce small variations of b_{ij} and small variations of the variables $W_1,\,\ldots\,,\,W_n$. They can be expressed by the formulas (4.9) and (4.10), though the functions $\tilde{h}_{ij}(x^0,x^1,x^2,x^3)$ and $\tilde{h}_i(x^0,x^1,x^2,x^3)$ now are determined by the functions $h^i(x^0,x^1,x^2,x^3)$ in (4.14). Applying (4.14) to the integral (4.3) with the function $\mathcal{H}$ written as (4.5), we get

$$\hat{H}=H+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{H}}{\delta P^i}\Bigr)_{\mathbf{Q},\mathbf{g},\boldsymbol{\beta}}h^i\,\sqrt{\det g}\,d^3x+\ldots. \tag{4.15}$$

Applying (4.14), (4.9), and (4.10) to the same integral (4.3) with the function $\mathcal{H}$ written as (4.4) and taking into account (3.7) and (3.8), we derive

$$\begin{gathered}\hat{H}=H+\varepsilon\int\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,\tilde{h}_{ij}+\sum_{i=1}^{n}\Bigl(W_i\,h^i+P^i\,\tilde{h}_i\Bigr)\Bigr)\sqrt{\det g}\,d^3x\,-\\ -\,\varepsilon\int\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\tilde{h}_{ij}+\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\tilde{h}_i\Bigr)\sqrt{\det g}\,d^3x+\ldots.\end{gathered} \tag{4.16}$$

If we take into account the relationships (3.25), then the formula (4.16) reduces to

$$\hat{H}=H+\varepsilon\int\sum_{j=n}^{3}W_i\,h^i\,\sqrt{\det g}\,d^3x+\ldots. \tag{4.17}$$

Comparing (4.17) with (4.15) as $\varepsilon\to 0$, we find that the second formula (4.7) is proved. Thus Theorem 4.1 is completely proved. □

5. Euler-Lagrange equations and Hamilton equations.

Let's return back to the formula (3.7). It is a short presentation of the formula (3.2). Therefore the action integral (1.12) now can be written as

$$S=\iint\mathcal{L}\,\sqrt{\det g}\,d^3x\,dx^0. \tag{5.1}$$

The arguments of the function $\mathcal{L}$ in (5.1) are shown in (3.8). When applying the stationary-action principle to the integral (5.1) the functions $b_{ij}(x^0,x^1,x^2,x^3)$ and

$g_{ij}(x^0,x^1,x^2,x^3)$ are not treated as independent parameters any more. They are related to each other through the formula (1.7). The same is true for the functions $W_i(x^0,x^1,x^2,x^3)$ and $Q_i(x^0,x^1,x^2,x^3)$. They are related to each other through the formula (3.4). Nevertheless the partial variational derivatives (3.17) are defined and the formulas (3.10), (3.12), (3.14), and (3.16) defining them can be used.

In order to apply the stationary-action principle with respect to the dynamical variables of matter $Q_1, \ldots, Q_n$ we introduce small variations to them:

$$\hat{Q}_i = Q_i(x^0,x^1,x^2,x^3) + \varepsilon\, h_i(x^0,x^1,x^2,x^3). \tag{5.2}$$

The formula (5.2) is similar to (3.11). However, unlike (3.11), now we take into account small variations of $W_1, \ldots, W_n$ induced by (5.2) due to the formula (3.4):

$$\hat{W}_i = W_i(x^0,x^1,x^2,x^3) + \varepsilon\,\frac{\partial h_i(x^0,x^1,x^2,x^3)}{\partial x^0}. \tag{5.3}$$

Applying (5.2) and (5.3) to (5.1) and using (3.10) and (3.12), we derive

$$\begin{aligned}\hat{S} = S + \varepsilon \iint \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\frac{\partial h_i}{\partial x^0}\sqrt{\det g}\;d^3x\,+\\ +\,\varepsilon\int\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}} h_i\sqrt{\det g}\;d^3x + \ldots\,.\end{aligned} \tag{5.4}$$

Integrating by parts in the first integral, we transform (5.4) as

$$\begin{aligned}\hat{S} = S - \varepsilon \iint \sum_{i=1}^{n}\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} h_i\sqrt{\det g}\;d^3x\,-\\ -\,\varepsilon\iint\sum_{i=1}^{n}\sum_{q=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} b^q_q\,h_i\sqrt{\det g}\;d^3x\,+\\ +\,\varepsilon\int\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}} h_i\sqrt{\det g}\;d^3x + \ldots\,.\end{aligned} \tag{5.5}$$

Since $h_i(x^0,x^1,x^2,x^3)$ are arbitrary smooth functions with compact support, from (5.5) we derive the following differential equation:

$$-\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} - \Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}} = 0, \tag{5.6}$$

where $i = 1, \ldots, n$. Note that $\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}$, where $\mathcal{L}_{\text{gr}}$ does not depend on Q_i and W_i (see (3.7) and (3.6)). Therefore the equations (5.6) are rewritten as

$$-\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} - \Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}} = 0. \tag{5.7}$$

Definition 5.1. The equations (5.7) are known as the Euler-Lagrange equations with respect to the dynamical variables $Q_1, \ldots, Q_n$ of matter.

Now let's apply the stationary-action principle with respect to the dynamical variables g_{ij} of the gravitational field. For this purpose we introduce small variations to the metric components g_{ij}:

$$\hat{g}_{ij} = g_{ij}(x^0, x^1, x^2, x^3) + \varepsilon\, h_{ij}(x^0, x^1, x^2, x^3). \tag{5.8}$$

The formula (5.8) is similar to (3.13). But unlike (3.13), now we take into account small variations of b_{ij} induced by (5.8) due to the formula (1.7):

$$\hat{b}_{ij} = b_{ij}(x^0, x^1, x^2, x^3) + \frac{\varepsilon}{2}\,\frac{\partial h_{ij}(x^0, x^1, x^2, x^3)}{\partial x^0}. \tag{5.9}$$

Applying (5.8) and (5.9) to (5.1) and using (3.14) and (3.16), we derive

$$\begin{aligned}\hat{S} = S + \frac{\varepsilon}{2}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\,\frac{\partial h_{ij}}{\partial x^0}\,\sqrt{\det g}\;d^3x\,+\\ +\,\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{aligned} \tag{5.10}$$

Integrating by parts in the first integral, we transform (5.10) as

$$\begin{aligned}\hat{S} = S - \frac{\varepsilon}{2}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\,h_{ij}\,\sqrt{\det g}\;d^3x\,-\\ -\frac{\varepsilon}{2}\iint\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{q=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\,b^q_q\;h_{ij}\,\sqrt{\det g}\;d^3x\,+\\ +\,\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{aligned} \tag{5.11}$$

Since $h_{ij}(x^0, x^1, x^2, x^3)$ are arbitrary smooth functions with compact support, from (5.11) we derive the following differential equation:

$$-\frac{1}{2}\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} - \frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}} = 0. \tag{5.12}$$

Since $\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}$, if we recall the formula (3.20), then we can rewrite (5.12) as

$$-\frac{1}{2}\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} - \frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}} = -\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}. \tag{5.13}$$

Definition 5.2. The equations (5.13) are known as the Euler-Lagrange equations with respect to the dynamical variables g_{ij} of the gravitational field.

Let's recall the energy function (4.2) and write it as follows:

$$H=\int\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\sum_{i=1}^{n}P^i\,W_i-\mathcal{L}\Bigr)\sqrt{\det g}\;d^3x. \tag{5.14}$$

The integral (4.3) with the function (4.5) is another presentation of (5.14). Keeping β^{ij} and P^i unchanged, we introduce small variations to g_{ij} and Q_i:

$$\hat g_{ij}=g_{ij}(x^0,x^1,x^2,x^3)+\varepsilon\,h_{ij}(x^0,x^1,x^2,x^3), \tag{5.15}$$

$$\hat Q_i=Q_i(x^0,x^1,x^2,x^3)+\varepsilon\,h_i(x^0,x^1,x^2,x^3). \tag{5.16}$$

Invertibility of the Legendre transformation (3.25) means that the variations (5.15) and (5.16) induce small variations of b_{ij} and small variations of the variables $W_1,\,\ldots,\,W_n$. They can be expressed by the formulas (4.9) and (4.10) where the functions $\tilde h_{ij}(x^0,x^1,x^2,x^3)$ and $\tilde h_i(x^0,x^1,x^2,x^3)$ are determined by the functions $h_{ij}(x^0,x^1,x^2,x^3)$ and $h_i(x^0,x^1,x^2,x^3)$ from (5.15) and (5.16). Applying (5.15) and (5.16) to the integral (4.3) with the function $\mathcal{H}$ written as (4.5), we derive

$$\begin{aligned}\hat H=H&+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{H}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\boldsymbol{\beta}}h_{ij}\sqrt{\det g}\;d^3x+\\&+\varepsilon\int\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{H}}{\delta Q_i}\Bigr)_{\mathbf{P},\mathbf{g},\boldsymbol{\beta}}h_i\sqrt{\det g}\;d^3x+\ldots.\end{aligned} \tag{5.17}$$

Similarly, applying (5.15) and (5.16) along with (4.9) and (4.10) to (5.14), we get

$$\begin{aligned}\hat H=H+\varepsilon\int\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,\tilde h_{ij}+\sum_{i=1}^{n}P^i\,\tilde h_i\Bigr)\sqrt{\det g}\;d^3x+\\+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\sum_{k=1}^{3}\sum_{q=1}^{3}\beta^{kq}\,b_{kq}+\sum_{k=1}^{n}P^k\,W_k\Bigr)\frac{g^{ij}\,h_{ij}}{2}\sqrt{\det g}\;d^3x-\\-\varepsilon\int\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\tilde h_{ij}+\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\tilde h_i\Bigr)\sqrt{\det g}\;d^3x-\\-\varepsilon\int\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}h_{ij}+\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}}h_i\Bigr)\sqrt{\det g}\;d^3x+\ldots.\end{aligned} \tag{5.18}$$

Taking into account the formulas (3.25), we can reduce (5.18) to

$$\begin{aligned}\hat H=H-\varepsilon\int\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}}h_i\sqrt{\det g}\;d^3x+\varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\sum_{k=1}^{n}\frac{P^k\,W_k}{2}\,g^{ij}+\\+\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{\beta^{kq}\,b_{kq}}{2}\,g^{ij}-\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}\Bigr)h_{ij}\sqrt{\det g}\;d^3x+\ldots.\end{aligned} \tag{5.19}$$

Now we can compare (5.17) and (5.19) as $\varepsilon \to 0$. As a result we derive

$$\begin{aligned}
&\Bigl(\frac{\delta\mathcal{H}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\boldsymbol{\beta}}=\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{\beta^{kq}\,b_{kq}}{2}\,g^{ij}+\sum_{k=1}^{n}\frac{P^k\,W_k}{2}\,g^{ij}-\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}},\\
&\Bigl(\frac{\delta\mathcal{H}}{\delta Q_i}\Bigr)_{\mathbf{P},\mathbf{g},\boldsymbol{\beta}}=-\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}}.
\end{aligned}\tag{5.20}$$

The relationships (5.20) are complementary to (3.25). Applying (3.25), (5.20), and (4.7) to (5.12) and (5.6), we derive the following pair of differential equations:

$$\begin{aligned}
&\frac{1}{2}\,\frac{\partial\beta^{ij}}{\partial x^0}=-\Bigl(\frac{\delta\mathcal{H}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\boldsymbol{\beta}}-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\Bigl(\frac{\delta\mathcal{H}}{\delta\beta^{kq}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}}\beta^{ij}+\\
&\qquad+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\beta^{kq}\Bigl(\frac{\delta\mathcal{H}}{\delta\beta^{kq}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}}g^{ij}+\frac{1}{2}\sum_{k=1}^{n}P^k\Bigl(\frac{\delta\mathcal{H}}{\delta P^k}\Bigr)_{\mathbf{Q},\mathbf{g},\boldsymbol{\beta}}g^{ij},\\
&\frac{\partial P^i}{\partial x^0}=-\Bigl(\frac{\delta\mathcal{H}}{\delta Q_i}\Bigr)_{\mathbf{P},\mathbf{g},\boldsymbol{\beta}}-\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\Bigl(\frac{\delta\mathcal{H}}{\delta\beta^{kq}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}}P^i.
\end{aligned}\tag{5.21}$$

Another pair of differential equations are derived from (4.7). Indeed, applying (1.7) and (3.4) to the left hand sides of the relationships (4.7), we get

$$\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}=\Bigl(\frac{\delta\mathcal{H}}{\delta\beta^{ij}}\Bigr)_{\mathbf{Q},\mathbf{P},\mathbf{g}},\qquad\qquad\frac{\partial Q_i}{\partial x^0}=\Bigl(\frac{\delta\mathcal{H}}{\delta P^i}\Bigr)_{\mathbf{Q},\mathbf{g},\boldsymbol{\beta}}.\tag{5.22}$$

Definition 5.3. The equations (5.21) and (5.22) constitute the system of Hamilton equations for the gravitational field and for matter.

The Hamilton equations (5.21) and (5.22) are equivalent to the Euler-Lagrange equations (5.6) and (5.12), though they are written with respect to a different set of dynamic variables.

6. Some explicit calculations.

Let's begin with the Euler-Lagrange equation (5.13). Its left hand side is determined by the function (3.6). We substitute this function for $\mathcal{L}$ into the integral (3.7) and then apply the small variation of $\mathbf{b}$ from (3.15) to this integral:

$$\begin{aligned}
\hat{L}_{\text{gr}}=L_{\text{gr}}+\frac{c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(&h_{ij}\,g^{ik}\,b_{kq}\,g^{jq}+b_{ij}\,g^{ik}\,h_{kq}\,g^{jq}-\\
&-h_{ij}\,g^{ij}\,b_{kq}\,g^{kq}-b_{ij}\,g^{ij}\,h_{kq}\,g^{kq}\bigr)\sqrt{\det g}\,d^3x+\ldots\,.
\end{aligned}\tag{6.1}$$

Due to the symmetry of g_{ij} and b_{ij} the formula (6.1) reduces to

$$\hat{L}_{\text{gr}}=L_{\text{gr}}+\frac{c^3}{8\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\biggl(b^{ij}-\sum_{k=1}^{3}b^k_k\,g^{ij}\biggr)h_{ij}\sqrt{\det g}\,d^3x+\ldots\,.\tag{6.2}$$

Comparing (6.2) with (3.16), we find

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\mathbf{Q},\mathbf{W},\mathbf{g}}=\frac{c^3}{8\pi\gamma}\left(b^{ij}-\sum_{k=1}^{3}b^k_k\,g^{ij}\right).\tag{6.3}$$

The second partial variational derivative in (5.13) is more complicated. In order to calculate it we need to substitute (3.6) for $\mathcal{L}$ into the integral (3.7) and then we need to apply the small variation (3.13) of the metric $\mathbf{g}$ to it. Upon substituting (3.6) into the integral (3.7) we subdivide this integral into three parts:

$$L_{\text{gr}}=L_1+L_2+L_3,\tag{6.4}$$

where

$$L_1=\frac{c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}b_{ij}\,g^{ik}\,b_{kq}\,g^{jq}\,\sqrt{\det g}\;d^3x,\tag{6.5}$$

$$L_2=-\frac{c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}b_{ij}\,g^{ij}\,b_{kq}\,g^{kq}\,\sqrt{\det g}\;d^3x,\tag{6.6}$$

$$L_3=\frac{c^3}{16\pi\gamma}\int(R-2\,\Lambda)\,\sqrt{\det g}\;d^3x.\tag{6.7}$$

From (3.13) we derive the following relationships:

$$\sqrt{\det\hat{g}}=\sqrt{\det g}\,\left(1+\varepsilon\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{g^{ij}\,h_{ij}}{2}\right)+\ldots\,,\tag{6.8}$$

$$\hat{g}^{ij}=g^{ij}-\varepsilon\sum_{p=1}^{3}\sum_{q=1}^{3}g^{ip}\,h_{pq}\,g^{qj}+\ldots\,.\tag{6.9}$$

Applying the relationships (6.8) and (6.9) to (6.5) and (6.6), we obtain

$$\begin{gathered}\hat{L}_1=L_1-\frac{\varepsilon\,c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}2\,b^i_k\,b^k_q\,g^{qj}\,-\right.\\ \left.-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g^{ij}\right)h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered}\tag{6.10}$$

$$\begin{gathered}\hat{L}_2=L_2+\frac{\varepsilon\,c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\sum_{k=1}^{3}2\,b^k_k\,b^{ij}\,-\right.\\ \left.-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g^{ij}\right)h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered}\tag{6.11}$$

In the case of the relationship (6.7) we have

$$\begin{gathered}\hat{L}_3=L_3-\frac{\varepsilon\,c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\left(R^{ij}-\frac{R}{2}\,g^{ij}+\Lambda\,g^{ij}\right)\cdot\\ \cdot\;h_{ij}\sqrt{\det g}\;d^3x\,dx^0+\ldots\,.\end{gathered}\tag{6.12}$$

The arguments and calculations supporting the formula (6.12) are the same as in deriving Einstein's gravity equation in § 2 of Chapter V in [6].

Now we substitute (6.10), (6.11), and (6.12) for L_1, L_2, and L_3 into the formula (6.4). As a result we derive the formula

$$\begin{aligned}\hat{L}_{\text{gr}} = L_{\text{gr}} + \frac{\varepsilon\,c^3}{16\pi\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3} 2\,b^i_k\,b^k_q\,g^{qj} +{}\\ + \sum_{k=1}^{3} 2\,b^k_k\,b^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g^{ij} - R^{ij} + \frac{R}{2}\,g^{ij} - \Lambda\,g^{ij}\Bigr)\,h_{ij}\,\sqrt{\det g}\;d^3x + \ldots\end{aligned}$$

that should be compared with the formula (3.14). This comparison yields

$$\begin{aligned}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}} = \frac{c^3}{16\pi\gamma}\Bigl(\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3} 2\,b^i_k\,b^k_q\,g^{qj} +{}\\ + \sum_{k=1}^{3} 2\,b^k_k\,b^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g^{ij} - R^{ij} + \frac{R}{2}\,g^{ij} - \Lambda\,g^{ij}\Bigr).\end{aligned} \tag{6.13}$$

The next step is to substitute (6.3) and (6.13) into the differential equation (5.13). Substituting (6.3) into (5.13), we get

$$-\frac{1}{2}\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} = \frac{c^3}{16\pi\gamma}\Bigl(-\frac{\partial b^{ij}}{\partial x^0} + \sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g^{ij} - \sum_{k=1}^{3} 2\,b^k_k\,b^{ij}\Bigr), \tag{6.14}$$

$$-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\sum_{q=1}^{3} b^q_q = \frac{c^3}{16\pi\gamma}\Bigl(\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\,b^q_q\,g^{ij} - \sum_{k=1}^{3} b^k_k\,b^{ij}\Bigr). \tag{6.15}$$

The expressions in the left hand sides of (6.13), (6.14), and (6.15) constitute the left hand side of the equation (5.13). Its right hand side is determined by the derivative (3.20). Applying (6.13), (6.14), and (6.15) to (5.13), we get

$$\begin{aligned}-\frac{\partial b^{ij}}{\partial x^0} + \sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g^{ij} + \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3} 2\,b^i_k\,b^k_q\,g^{qj} -{}\\ - \sum_{k=1}^{3} b^k_k\,b^{ij} + \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g^{ij} - R^{ij} + \frac{R}{2}\,g^{ij} - \Lambda\,g^{ij} = -\frac{16\pi\gamma}{c^3}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}.\end{aligned} \tag{6.16}$$

The equation (6.16) is similar to (1.4). In order to make these equations more similar we need to lower indices i and j in (6.16). Taking into account the relationship (3.21) and using symmetry of g_{ij} and b_{ij}, we derive

$$\begin{aligned}-\frac{\partial b_{ij}}{\partial x^0} + \sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij} + \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g_{ij} + \sum_{k=1}^{3}\sum_{q=1}^{3}(b_{ki}\,b^k_j + b_{kj}\,b^k_i) -{}\\ - \sum_{k=1}^{3} b^k_k\,b_{ij} + \sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g_{ij} - R_{ij} + \frac{R}{2}\,g_{ij} - \Lambda\,g_{ij} = \frac{16\pi\gamma}{c^3}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}.\end{aligned} \tag{6.17}$$

Now, comparing (6.17) and (1.4), we see that these two equations do coincide provided the relationship (1.13) is fulfilled. The relationship (1.13) was derived in [11]. Therefore we conclude that the formula (6.17) proves the following theorem.

Theorem 6.1. *The 3D Euler-Lagrange equation* (5.13) *is equivalent to the equation* (1.4) *that was derived from the four-dimensional Einstein's gravity equation* (1.3).

Proceeding to the Hamilton equations (5.21) and (5.22), we choose the special case determined by the condition (3.22), see also remark on page 5 and (3.5). In this case special case the relationship (3.20) reduces to (3.23). Note that $\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}$. The function $\mathcal{L}_{\text{gr}}$ is given by the explicit formula (3.6). It does not depend on $W_1, \ldots, W_n$ and $Q_1, \ldots, Q_n$. Therefore apart from (3.22) we have

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta W_i}\right)_{\mathbf{Q},\mathbf{g},\mathbf{b}} = 0, \qquad\qquad \left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta Q_i}\right)_{\mathbf{W},\mathbf{g},\mathbf{b}} = 0. \tag{6.18}$$

Applying (3.22) and (6.18) to (3.25), we obtain

$$\beta^{ij} = \left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\mathbf{g}}, \qquad\qquad P^i = \left(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta W_i}\right)_{\mathbf{Q},\mathbf{g}}. \tag{6.19}$$

Substituting (6.19) into (4.2) and taking into account $\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}$, we derive

$$\mathcal{H} = \mathcal{H}_{\text{gr}} + \mathcal{H}_{\text{mat}}, \tag{6.20}$$

i. e. the energy function H is subdivided into two parts responsible for the gravitational field and for matter. The functions H_{gr} and H_{mat} are given by the formulas

$$H_{\text{gr}} = \int\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij} - \mathcal{L}_{\text{gr}}\right)\sqrt{\det g}\;d^3x, \tag{6.21}$$

$$H_{\text{mat}} = \int\left(\sum_{i=1}^{n}P^i\,W_i - \mathcal{L}_{\text{mat}}\right)\sqrt{\det g}\;d^3x, \tag{6.22}$$

The partial variational derivative in the first formula (6.19) is already calculated in (6.3). Therefore we can write the following explicit formula:

$$\beta^{ij} = \frac{c^3}{8\pi\gamma}\left(b^{ij} - \sum_{k=1}^{3} b^k_k\,g^{ij}\right). \tag{6.23}$$

The relationship (6.23) is invertible. Its inverse is written as

$$b^{ij} = \frac{8\pi\gamma}{c^3}\left(\beta^{ij} - \frac{1}{2}\sum_{k=1}^{3}\beta^k_k\,g^{ij}\right). \tag{6.24}$$

We have no explicit presentation for the second formula (6.19). Therefore we shall just assume that it is invertible in the sense of Theorem 4.1. Applying this theorem to (6.21) and (6.22), we obtain the following two functions in (6.20):

$$\mathcal{H}_{\text{gr}} = \mathcal{H}_{\text{gr}}(\mathbf{g}, \boldsymbol{\beta}), \tag{6.25}$$

$$\mathcal{H}_{\text{mat}} = \mathcal{H}_{\text{mat}}(Q_1, \ldots, Q_n, P^1, \ldots, P^n, \mathbf{g}). \tag{6.26}$$

Substituting (6.25) and (6.26) into (6.20) and then substituting (6.20) into the equations (5.21) and (5.22), we derive the following Hamilton equations for matter:

$$\frac{\partial Q_i}{\partial x^0} = \Bigl(\frac{\delta \mathcal{H}_{\text{mat}}}{\delta P^i}\Bigr)_{\mathbf{Q},\mathbf{g}}, \tag{6.27}$$

$$\frac{\partial P^i}{\partial x^0} = -\Bigl(\frac{\delta \mathcal{H}_{\text{mat}}}{\delta Q_i}\Bigr)_{\mathbf{P},\mathbf{g},} - \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\Bigl(\frac{\delta \mathcal{H}_{\text{gr}}}{\delta \beta^{kq}}\Bigr)_{\mathbf{g}} P^i. \tag{6.28}$$

Similarly we derive the Hamilton equations for the gravitational field:

$$\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0} = \Bigl(\frac{\delta \mathcal{H}_{\text{gr}}}{\delta \beta^{ij}}\Bigr)_{\mathbf{g}}, \tag{6.29}$$

$$\begin{aligned} \frac{1}{2}\,\frac{\partial \beta^{ij}}{\partial x^0} &= -\Bigl(\frac{\delta \mathcal{H}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\boldsymbol{\beta}} - \frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\Bigl(\frac{\delta \mathcal{H}_{\text{gr}}}{\delta \beta^{kq}}\Bigr)_{\mathbf{g}} \beta^{ij} + \\ &+ \frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} \beta^{kq}\Bigl(\frac{\delta \mathcal{H}_{\text{gr}}}{\delta \beta^{kq}}\Bigr)_{\mathbf{g}} g^{ij} - \Bigl(\frac{\delta \mathcal{H}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P}} + \frac{1}{2}\sum_{k=1}^{n} P^k\Bigl(\frac{\delta \mathcal{H}_{\text{mat}}}{\delta P^k}\Bigr)_{\mathbf{Q},\mathbf{g}} g^{ij}. \end{aligned} \tag{6.30}$$

The last term in (6.28) and two last terms in (6.30) are directly responsible for matter to gravity interaction.

We shall not try to make more explicit the equations (6.27) and (6.28) because in this paper we do not specify the sorts of matter and the nature of the dynamic variables $Q_1, \ldots, Q_n$ of matter as well as their associated momenta $P_1, \ldots, P_n$. The equations (6.29) and (6.30) are different. In this case we can make explicit all terms except for the last two terms in (6.30). Let's begin with the Hamilton function (6.25). According to Definition 4.2 the Hamilton function is the density of the energy function expressed through the dynamic variables and their associated momenta. The energy function of the gravitational field is given by (6.21). Hence

$$\mathcal{H}_{\text{gr}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \beta^{ij}\, b_{ij} - \mathcal{L}_{\text{gr}}, \tag{6.31}$$

where $\mathcal{L}_{\text{gr}}$ is given by the formula (3.6). In order to replace the components of the tensor field $\mathbf{b}$ by the components of the tensor field $\boldsymbol{\beta}$ in (6.31) we use (6.24):

$$\mathcal{H}_{\text{gr}} = \frac{8\pi\gamma}{c^3}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(\beta^{kq}\,\beta_{kq} - \frac{1}{2}\beta^k_k\,\beta^q_q\Bigr) - \mathcal{L}_{\text{gr}}. \tag{6.32}$$

The function $\mathcal{L}_{\text{gr}}$ also comprises the entries of $\mathbf{b}$. As a result of applying (6.24) to $\mathcal{L}_{\text{gr}}$ in (3.6) we transform it as follows:

$$\mathcal{L}_{\text{gr}} = \frac{4\pi\gamma}{c^3}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(\beta^{kq}\,\beta_{kq} - \frac{1}{2}\beta^k_k\,\beta^q_q\Bigr) + \frac{c^3}{16\pi\gamma}(R - 2\,\Lambda). \tag{6.33}$$

Substituting (6.33) into (6.32) yields

$$\mathcal{H}_{\text{gr}} = \frac{4\pi\gamma}{c^3}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(\beta^{kq}\,\beta_{kq} - \frac{1}{2}\beta^k_k\,\beta^q_q\Bigr) - \frac{c^3}{16\pi\gamma}(R - 2\,\Lambda). \tag{6.34}$$

The energy function H_{gr} is produced from (6.34) by integration:

$$
\begin{gathered}
H_{\text{gr}}=\frac{4\pi\gamma}{c^3}\int\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(\beta^{kq}\,\beta_{kq}-\frac{1}{2}\beta^k_k\,\beta^q_q\Bigr)\,\sqrt{\det g}\;d^3x\,-\\
-\,\frac{c^3}{16\pi\gamma}\int(R-2\,\Lambda)\,\sqrt{\det g}\;d^3x.
\end{gathered}\tag{6.35}
$$

The integrals in (6.35) can be used in order to calculate partial variational derivatives of the function $\mathcal{H}_{\text{gr}}$ in (6.34). The procedure is similar to that of (6.1) and (6.4), (6.5), (6.6), (6.7) with subsequent calculations. It yields

$$
\Bigl(\frac{\delta\mathcal{H}_{\text{gr}}}{\delta\beta^{ij}}\Bigr)_{\mathbf{g}}=\frac{8\pi\gamma}{c^3}\Bigl(\beta_{ij}-\frac{1}{2}\sum_{k=1}^{3}\beta^k_k\,g_{ij}\Bigr),\tag{6.36}
$$

$$
\begin{gathered}
\Bigl(\frac{\delta\mathcal{H}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\boldsymbol\beta}=\frac{4\pi\gamma}{c^3}\Bigl(\sum_{k=1}^{3}2\,\beta^i_k\,\beta^{kj}-\sum_{k=1}^{3}\beta^k_k\,\beta^{ij}+\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,\beta^k_q\,\beta^q_k\,g^{ij}\,-\\
-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{4}\beta^k_k\,\beta^q_q\,g^{ij}\Bigr)+\frac{c^3}{16\pi\gamma}\Bigl(R^{ij}-\frac{R}{2}\,g^{ij}+\Lambda\,g^{ij}\Bigr).
\end{gathered}\tag{6.37}
$$

Applying (6.36) to the second and third terms in the right hand side of (6.30) yields

$$
\begin{gathered}
-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\Bigl(\frac{\delta\mathcal{H}_{\text{gr}}}{\delta\beta^{kq}}\Bigr)_{\mathbf{g}}\,\beta^{ij}=\frac{2\pi\gamma}{c^3}\sum_{k=1}^{3}\beta^k_k\,\beta^{ij},\\
\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\beta^{kq}\Bigl(\frac{\delta\mathcal{H}_{\text{gr}}}{\delta\beta^{kq}}\Bigr)_{\mathbf{g}}\,g^{ij}=\frac{4\pi\gamma}{c^3}\Bigl(\sum_{k=1}^{3}\sum_{q=1}^{3}\beta^k_q\,\beta^q_k\,g^{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\beta^k_k\,\beta^q_q\,g^{ij}\Bigr).
\end{gathered}\tag{6.38}
$$

Now we substitute (6.36), (6.37), and (6.38) into the equations (6.29) and (6.30). As a result we obtain the explicit form of the Hamilton equations for gravity:

$$
\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}=\frac{8\pi\gamma}{c^3}\Bigl(\beta_{ij}-\frac{1}{2}\sum_{k=1}^{3}\beta^k_k\,g_{ij}\Bigr),\tag{6.39}
$$

$$
\begin{gathered}
\frac{1}{2}\,\frac{\partial\beta^{ij}}{\partial x^0}=\frac{4\pi\gamma}{c^3}\Bigl(\sum_{k=1}^{3}\frac{3}{2}\,\beta^k_k\,\beta^{ij}-\sum_{k=1}^{3}2\,\beta^i_k\,\beta^{kj}+\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,\beta^k_q\,\beta^q_k\,g^{ij}\,-\\
-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{4}\beta^k_k\,\beta^q_q\,g^{ij}\Bigr)-\frac{c^3}{16\pi\gamma}\Bigl(R^{ij}-\frac{R}{2}\,g^{ij}+\Lambda\,g^{ij}\Bigr)\,-\\
-\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P}}+\frac{1}{2}\sum_{k=1}^{n}P^k\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta P^k}\Bigr)_{\mathbf{Q},\mathbf{g}}\,g^{ij}.
\end{gathered}\tag{6.40}
$$

Due to (1.7) the equation (6.39) is equivalent to (6.24). In order to compare (6.40) with (6.16) and (1.4) we need to rewrite it in terms of the components of the tensor field $\mathbf{b}$. For this purpose we use (6.23). Differentiating (6.23), we get

$$
\frac{1}{2}\,\frac{\partial\beta^{ij}}{\partial x^0}=\frac{c^3}{16\pi\gamma}\Bigl(\frac{\partial b^{ij}}{\partial x^0}-\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g^{ij}+\sum_{k=1}^{3}2\,b^k_k\,b^{ij}\Bigr).\tag{6.41}
$$

Now, applying (6.41) and (6.23) to (6.40), we derive

$$\begin{aligned}&\frac{\partial b^{ij}}{\partial x^0}-\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g^{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_q\,b^q_k\,g^{ij}+\sum_{k=1}^{3}2\,b^i_k\,b^{kj}+\\&+\sum_{k=1}^{3}b^k_k\,b^{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{1}{2}\,b^k_k\,b^q_q\,g^{ij}+R^{ij}-\frac{R}{2}\,g^{ij}+\Lambda\,g^{ij}=\\&=\frac{16\pi\gamma}{c^3}\,\biggl(-\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P}}+\frac{1}{2}\sum_{k=1}^{n}P^k\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta P^k}\Bigr)_{\mathbf{Q},\mathbf{g}}\,g^{ij}\biggr).\end{aligned}\tag{6.42}$$

Comparing (6.42) with (6.16), we see that these two equations are equivalent to each other provided the following equality is fulfilled:

$$-\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P}}+\frac{1}{2}\sum_{k=1}^{n}P^k\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta P^k}\Bigr)_{\mathbf{Q},\mathbf{g}}\,g^{ij}=\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\tag{6.43}$$

The right hand side of the equality (6.43) is given by the formula (3.20). Let's recall that we have derived the equation (6.42) from the equation (6.40), while the equations (6.39) and (6.40) were derived under the assumption that the relationships (3.22) and (6.18) are fulfilled. In this case the formula (3.20) reduces to (3.23), while the Legendre transformation (3.25) is written as (6.19). Applying the formula (3.23) to (6.43), we transform the relationship (6.43) as

$$-\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{Q},\mathbf{P}}+\frac{1}{2}\sum_{k=1}^{n}P^k\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta P^k}\Bigr)_{\mathbf{Q},\mathbf{g}}\,g^{ij}=\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q}}.\tag{6.44}$$

The relationship (6.44) is similar to (5.20). It is derived in a way similar to that of (5.20). Let's recall the formula (6.22) and complement it with the formula

$$H_{\text{mat}}=\int\mathcal{H}_{\text{mat}}\,\sqrt{\det g}\;d^3x,\tag{6.45}$$

Keeping the variables $Q_1,\,\ldots\,,\,Q_n$ and $P^1,\,\ldots\,,\,P^n$ unchanged, we introduce small variations to the components of metric g_{ij}:

$$\hat{g}_{ij}=g_{ij}(x^0,x^1,x^2,x^3)+\varepsilon\,h_{ij}(x^0,x^1,x^2,x^3),\tag{6.46}$$

The variations (6.46) induce small variations of the variables $W_1,\,\ldots\,,\,W_n$:

$$\hat{W}_i=W_i(x^0,x^1,x^2,x^3)+\varepsilon\,\tilde{h}_i(x^0,x^1,x^2,x^3).\tag{6.47}$$

Applying both (6.46) and (6.47) to (6.22), we derive

$$\begin{aligned}\hat{H}_{\text{mat}}&=H_{\text{mat}}+\varepsilon\int\sum_{k=1}^{n}\biggl(P^k\,\tilde{h}_k+\sum_{i=1}^{3}\sum_{j=1}^{3}P^k\,W_k\,\frac{g^{ij}\,h_{ij}}{2}\biggr)\sqrt{\det g}\;d^3x\,-\\&-\varepsilon\int\biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q}}h_{ij}+\sum_{k=1}^{n}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta W_k}\Bigr)_{\mathbf{Q},\mathbf{g}}\tilde{h}_k\biggr)\sqrt{\det g}\;d^3x+\ldots.\end{aligned}\tag{6.48}$$

Similarly, applying (6.46) to the integral (6.45), we obtain

$$\hat{H}_{\text{mat}} = H_{\text{mat}} + \varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{P},\mathbf{Q}} h_{ij}\,\sqrt{\det g}\;d^3x + \ldots . \tag{6.49}$$

Due to the second relationship in (6.19) the formula (6.48) simplifies as follows:

$$\begin{aligned}\hat{H}_{\text{mat}} = H_{\text{mat}} + \varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{1}{2}\sum_{k=1}^{n} P^k\,W_k\,g^{ij}\Bigr)\,h_{ij}\,\sqrt{\det g}\;d^3x\,-\\ -\,\varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q}} h_{ij}\,\sqrt{\det g}\;d^3x + \ldots .\end{aligned} \tag{6.50}$$

The inversion of the Legendre transformation given by the relationships (6.19) is determined by Theorem 4.1 and the relationships (4.7) therein. Due to (6.20) and since $\mathcal{H}_{\text{gr}}$ in (6.34) does not depend on $P^1,\,\ldots\,,\,P^n$ and $Q_1,\,\ldots\,,\,Q_n$, while $\mathcal{H}_{\text{mat}}$ does not depend on β^{ij}, the relationships (4.7) in our present case are written as

$$b_{ij} = \Bigl(\frac{\delta\mathcal{H}_{\text{gr}}}{\delta\beta^{ij}}\Bigr)_{\mathbf{g}}, \qquad\qquad W_i = \Bigl(\frac{\delta\mathcal{H}_{\text{mat}}}{\delta P^i}\Bigr)_{\mathbf{Q},\mathbf{g}}. \tag{6.51}$$

Now the formula (6.44) is derived by substituting the second relationship (6.51) into (6.50) and comparing the formulas (6.49) and (6.50) as $\varepsilon \to 0$.

Having proved (6.44) and hence having proved the relationship (6.43), we conclude that the equation (6.40) is equivalent to the equations (6.16) and (6.17). The equation (6.17) coincides with the equation (1.4) due to (1.13). This means that we have proved the following theorem.

Theorem 6.2. *In the special case determined by the relationship* (3.22) *the Hamilton equations* (6.29) *and* (6.30) *are equivalent to the equations* (1.7) *and* (1.4) *through the direct and inverse Legendre transformations given by* (6.19) *and* (6.51).

7. Concluding remarks.

The main result of the present paper is the Hamiltonian approach applied to describing the dynamics of the gravitational field and matter within the paradigm of a 3D-brane universe from [1]. In general case this result is expressed by the Hamilton equations (5.21) and (5.22) which are equivalent to the Euler-Lagrange equations (5.6) and (5.12). Due to the subdivision $\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}$ the equations (5.6) and (5.12) are written as (5.7) and (5.13). It turns out that the Euler-Lagrange equation of the gravitational field (5.13) is equivalent to the equation (1.4) that was previously derived in [1] and [11].

The subdivision $\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}$ leads to the subdivision $\mathcal{H} = \mathcal{H}_{\text{gr}} + \mathcal{H}_{\text{mat}}$ of the function (4.4) through the formulas (6.21) and (6.22). However in general case this subdivision does not imply separation of dynamic variables, i. e. the dynamic variables of matter can enter the function $\mathcal{H}_{\text{gr}}$. Therefore the special case given by the condition (3.22) was considered. In this special case the function $\mathcal{H}_{\text{gr}}$ does not comprise the dynamic variables of matter and is given by the explicit formula

(6.34). It turns out that the Hamilton equations (6.29) and (6.30) written with the use of this function are equivalent to the equations (1.7) and (1.4).

Although the Hamiltonian approach developed above in this paper does not lead to equations other than those previously derived in [1] and [11], it is important from the conceptual point of view. It can be helpful for quantization of the gravitational field within the framework of our 3D-brane paradigm.

8. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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