

# ENERGY CONSERVATION LAW FOR THE GRAVITATIONAL FIELD IN A 3D-BRANE UNIVERSE.

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ABSTRACT. The 3D-brane universe model is an alternative theory of gravity. It was introduced using the concept of temporal coexistence. In the present paper the energy conservation law for the gravitational field and matter in the 3D-brane universe model is derived.

## 1. INTRODUCTION.

The concept of temporal coexistence was suggested in [1] (see also [2] and [3]). It applies to the events of the four-dimensional spacetime and subdivides them into those that had happened in the past, those happening right now, and forthcoming events that did not yet happen. This subdivision is global and it is absolute, i. e. it is irrespective to the choice of a coordinate frame and to the instrumentation used for measuring time. The events happening right now are called currently coexisting. They constitute a 3D-brane representing the current state of a universe in question.

Within the paradigm of a 3D-brane universe the gravitational field is described by a time-dependent 3D metric with the components

$$g_{ij} = g_{ij}(x^0, x^1, x^2, x^3), \quad 1 \leq i, j \leq 3, \quad (1.1)$$

where  $x^0 = ct$  and  $c$  is the speed of light. We relate this 3D metric (1.1) to the four-dimensional metric of the standard 4D paradigm through the formulas

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -g_{11} & -g_{12} & -g_{13} \\ 0 & -g_{21} & -g_{22} & -g_{23} \\ 0 & -g_{31} & -g_{32} & -g_{33} \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -g^{11} & -g^{12} & -g^{13} \\ 0 & -g^{21} & -g^{22} & -g^{23} \\ 0 & -g^{31} & -g^{32} & -g^{33} \end{pmatrix}. \quad (1.2)$$

If the formulas (1.2) are fulfilled, then in the standard paradigm the time variable  $t$  in  $x^0 = ct$  is interpreted as the cosmological time (see [4]), while  $x^1, x^2, x^3$  are interpreted as comoving spatial coordinates (see [5]).

In the standard four-dimensional paradigm of general relativity and cosmology the four-dimensional metric should obey the standard Einstein's equation

$$r_{ij} - \frac{r}{2} G_{ij} - \Lambda G_{ij} = \frac{8\pi\gamma}{c^4} T_{ij}, \quad (1.3)$$

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2010 *Mathematics Subject Classification.* 83D05, 83C40, 83F05, 70H03, 53B50, 53Z05.

*Key words and phrases.* 3D-brane universe, energy conservation law.

see §2 in Chapter V of [6]. Here  $c$  is the speed of light,  $\gamma$  is Newton's gravitational constant (see [7]), and  $\Lambda$  is the cosmological constant (see [8]). The quantities  $T_{ij}$  in the right hand side of (1.3) are the components of the energy-momentum tensor (see [9]). The term  $r_{ij}$  in (1.3) corresponds to the components of the four-dimensional Ricci tensor and  $r$  is the four-dimensional scalar curvature (see §8 in Chapter IV of [10]). By substituting (1.2) into (1.3) in [1] the following equations were derived:

$$\begin{aligned} & \frac{\partial b_{ij}}{\partial x^0} - \sum_{k=1}^3 \frac{\partial b_k^k}{\partial x^0} g_{ij} - \sum_{k=1}^3 (b_{ki} b_j^k + b_{kj} b_i^k) - \frac{g_{ij}}{2} \sum_{k=1}^3 \sum_{q=1}^3 b_q^k b_k^q - \\ & - \frac{g_{ij}}{2} \sum_{k=1}^3 \sum_{q=1}^3 b_k^k b_q^q + \sum_{k=1}^3 b_k^k b_{ij} + R_{ij} - \frac{R}{2} g_{ij} + \Lambda g_{ij} = \frac{8\pi\gamma}{c^4} T_{ij}, \end{aligned} \quad (1.4)$$

$$\sum_{k=1}^3 \nabla_k b_j^k - \sum_{k=1}^3 \nabla_j b_k^k = \frac{8\pi\gamma}{c^4} T_{0j}, \quad (1.5)$$

$$-\frac{1}{2} \sum_{k=1}^3 \sum_{q=1}^3 b_q^k b_k^q + \frac{1}{2} \sum_{k=1}^3 \sum_{q=1}^3 b_k^k b_q^q + \frac{R}{2} - \Lambda = \frac{8\pi\gamma}{c^4} T_{00}. \quad (1.6)$$

Here  $R_{ij}$  are the components of the three-dimensional Ricci tensor,  $R$  is the three-dimensional scalar curvature, and  $b_{ij}$  are given by the formula

$$b_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^0} = \frac{\dot{g}_{ij}}{2c}. \quad (1.7)$$

In [11] the Lagrangian approach to the 3D-brane universe model was applied. The action integral was taken as a sum of two integrals:

$$S = \iint \mathcal{L}_{\text{gr}} \sqrt{\det g} d^3x dx^0 + \iint \mathcal{L}_{\text{mat}} \sqrt{\det g} d^3x dx^0. \quad (1.8)$$

The first integral in (1.8) corresponds to the gravitational field, while the second one is responsible for matter. The Lagrangian  $\mathcal{L}_{\text{gr}}$  was taken in the standard four-dimensional form (see §2 in Chapter V of [6]):

$$\mathcal{L}_{\text{gr}} = -\frac{c^3}{16\pi\gamma} (r + 2\Lambda). \quad (1.9)$$

The four-dimensional scalar curvature  $r$  in (1.9) is associated with the four-dimensional metric (1.2). As is was shown in [1], it is expressed through the three-dimensional scalar curvature  $R$  in the following way:

$$r = -2 \sum_{k=1}^3 \frac{\partial b_k^k}{\partial x^0} - R - \sum_{k=1}^3 \sum_{q=1}^3 b_q^k b_k^q - \sum_{k=1}^3 \sum_{q=1}^3 b_k^k b_q^q. \quad (1.10)$$

Applying the stationary-action principle to (1.8), in [11] the equation (1.4) was re-derived along with the following purely three-dimensional expression for the spatial components of the four-dimensional energy-momentum tensor in it:

$$T_{ij} = -2c \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{ij}}, \quad 1 \leq i, j \leq 3. \quad (1.11)$$

In [12] the Lagrangian approach to the 3D-brane universe model was considered again and the Hamiltonian approach to this model was developed. As a result the equation (1.4) was rederived twice more. As for the equations (1.5) and (1.6), they did not arise neither in [11] nor in [12]. The main goal of the present paper is to formulate and prove the energy conservation law for the gravitational field and matter within the 3D-brane universe paradigm.

## 2. REDUCING THE ORDER OF THE LAGRANGIAN.

Due to (1.7) the Lagrangian (1.9) with  $r$  given by (1.10) is of the second order with respect to time derivatives of the metric (1.1). In [12] the order of this Lagrangian was reduced by one. For this purpose in [12] the function (1.10) was replaced by the following function:

$$\rho = \sum_{k=1}^3 \sum_{q=1}^3 b_k^k b_q^q - R - \sum_{k=1}^3 \sum_{q=1}^3 b_q^k b_k^q. \quad (2.1)$$

Then the Lagrangian (1.9) was rewritten as

$$\mathcal{L}_{\text{gr}} = -\frac{c^3}{16\pi\gamma} (\rho + 2\Lambda). \quad (2.2)$$

The action integral (1.8) with the Lagrangian (2.2), where  $\rho$  is given by the formula (2.1), leads to the same differential equation (1.4) as the action integral (1.8) with the Lagrangian (1.9). The Lagrangian (2.2) is of the first order with respect to time derivatives of the metric (1.1).

## 3. EULER-LAGRANGE EQUATIONS.

The Lagrangian of matter  $\mathcal{L}_{\text{mat}}$  was not written in full details in [11]. This was done later in [12]. Following [12], we write

$$\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}}(Q_1, \dots, Q_n, W_1, \dots, W_n, \mathbf{g}, \mathbf{b}). \quad (3.1)$$

Here  $Q_1, \dots, Q_n$  are dynamic variables of matter and  $W_1, \dots, W_n$  are their time derivatives. They are related to each other through the formula

$$W_i = \frac{\partial Q_i}{\partial x^0} = \frac{\dot{Q}_i}{c}, \quad i = 1, \dots, n. \quad (3.2)$$

The quantities  $Q_1, \dots, Q_n$  and  $W_1, \dots, W_n$  are fields, i.e. they depend on the spatial variables  $x^1, x^2, x^3$  and on the temporal variable  $x^0 = ct$ :

$$Q_i = Q_i(x^0, x^1, x^2, x^3), \quad W_i = W_i(x^0, x^1, x^2, x^3). \quad (3.3)$$

The same is true for the components  $g_{ij}$  and  $b_{ij}$  of the fields  $\mathbf{g}$  and  $\mathbf{b}$  in (3.1). The formula (3.2) here is analogous to the formula (1.7).

The Lagrangian  $\mathcal{L}_{\text{gr}}$  in (2.2) does not depend on the dynamic variables of matter  $Q_1, \dots, Q_n$  and  $W_1, \dots, W_n$ . Therefore we write

$$\mathcal{L}_{\text{gr}} = \mathcal{L}_{\text{gr}}(\mathbf{g}, \mathbf{b}). \quad (3.4)$$

The total Lagrangian  $\mathcal{L}$  in the action integral (1.8) is the sum

$$\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}} \quad (3.5)$$

of the Lagrangians (3.1) and (3.4). Therefore in this case we write

$$\mathcal{L} = \mathcal{L}(Q_1, \dots, Q_n, W_1, \dots, W_n, \mathbf{g}, \mathbf{b}). \quad (3.6)$$

**Remark.** Each entry  $Q_i$  in the argument list in (3.1) or in (3.6) means that the corresponding Lagrangian depends not only on  $Q_i$ , but on some finite number of partial derivatives of  $Q_i$  with respect to spatial variables  $x^1, x^2, x^3$ . The same is true for the arguments  $W_i$  and for the components  $g_{ij}$  and  $b_{ij}$  of the tensor fields  $\mathbf{g}$  and  $\mathbf{b}$  in (3.1), (3.4), and (3.6).

Relying on (3.3) and on the above remark, we introduce the following notations:

$$Q_i[i_1 \dots i_s] = \frac{\partial Q_i}{\partial x^{i_1} \dots \partial x^{i_s}}, \quad W_i[i_1 \dots i_s] = \frac{\partial W_i}{\partial x^{i_1} \dots \partial x^{i_s}}. \quad (3.7)$$

Then we introduce the notations similar to (3.7) for the components of  $\mathbf{g}$  and  $\mathbf{b}$ :

$$g_{ij}[i_1 \dots i_s] = \frac{\partial g_{ij}}{\partial x^{i_1} \dots \partial x^{i_s}}, \quad b_{ij}[i_1 \dots i_s] = \frac{\partial b_{ij}}{\partial x^{i_1} \dots \partial x^{i_s}}. \quad (3.8)$$

In terms of the Lagrangian (3.5) the action integral (1.8) is written as

$$S = \iint \mathcal{L} \sqrt{\det g} \, d^3x \, dx^0. \quad (3.9)$$

Applying the stationary action principle to the action integral (3.9), in [12] the following Euler-Lagrange equations were derived:

$$-\frac{1}{2} \frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} - \frac{1}{2} \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} \sum_{q=1}^3 b_q^q + \left( \frac{\delta \mathcal{L}}{\delta g_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{b}} = 0, \quad (3.10)$$

$$-\frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} - \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} \sum_{q=1}^3 b_q^q + \left( \frac{\delta \mathcal{L}}{\delta Q_i} \right)_{\mathbf{w}, \mathbf{g}, \mathbf{b}} = 0. \quad (3.11)$$

In [12] it was shown that the equation (3.10) is equivalent to the equation (1.4) provided we admit the equality (1.11). For the variational derivative in the right hand side of (1.11) in [12] the following formulas were derived:

$$\begin{aligned} \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g^{ij}} &= - \sum_{k=1}^3 \sum_{q=1}^3 \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g_{kq}} g_{ik} g_{qj}, \\ \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g_{ij}} &= - \frac{1}{2} \frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}_{\text{mat}}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} - \\ &\quad - \frac{1}{2} \left( \frac{\delta \mathcal{L}_{\text{mat}}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} \sum_{q=1}^3 b_q^q + \left( \frac{\delta \mathcal{L}_{\text{mat}}}{\delta g_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{b}}. \end{aligned}$$

As for the equation (3.11), it describes the dynamics of matter. Since  $\mathcal{L}_{\text{gr}}$  does not depend on  $Q_i$  and  $W_i$  (see (3.4), this equation can be rewritten as

$$-\frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}_{\text{mat}}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} - \left( \frac{\delta \mathcal{L}_{\text{mat}}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} \sum_{q=1}^3 b_q^q + \left( \frac{\delta \mathcal{L}_{\text{mat}}}{\delta Q_i} \right)_{\mathbf{W}, \mathbf{g}, \mathbf{b}} = 0.$$

#### 4. LEGENDRE TRANSFORMATION AND THE ENERGY DENSITY.

The Legendre transformation determined by the Lagrangian (3.5) of the gravitational field and matter is given by the following formulas:

$$\beta^{ij} = \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{W}, \mathbf{Q}, \mathbf{g}}, \quad P^i = \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}}. \quad (4.1)$$

Here  $\beta^{ij}$  and  $P^i$  are generalized momenta associated with the generalized velocities  $b_{ij}$  and  $W_i$ . The transformation (4.1) was introduced in [12].

The energy density is introduced by the following formula from [12]:

$$\mathcal{H} = \sum_{i=1}^3 \sum_{j=1}^3 \beta^{ij} b_{ij} + \sum_{i=1}^n P^i W_i - \mathcal{L}. \quad (4.2)$$

The quantities  $\beta^{ij}$  and  $P^i$  in (4.2) are given by the formulas (4.1). Let  $\Omega$  be a three-dimensional domain in a 3D-brane universe. The energy of the gravitational field and matter within this domain is given by the following integral:

$$E(\Omega) = \int_{\Omega} \mathcal{H} \sqrt{\det g} \, d^3 x. \quad (4.3)$$

The main goal of the next section is to derive a formula for the time derivative of the energy integral (4.3).

#### 5. THE ENERGY CONSERVATION LAW.

Let's consider a small increment of the variable  $x^0$ . We write it as follows:

$$\hat{x}^0 = x^0 + \varepsilon. \quad (5.1)$$

Then we apply (5.1) to the dynamic variables of the gravitational field and matter:

$$\hat{g}_{ij} = g_{ij}(\hat{x}^0, x^1, x^2, x^3), \quad \hat{Q}_i = Q_i(\hat{x}^0, x^1, x^2, x^3), \quad (5.2)$$

$$\hat{b}_{ij} = b_{ij}(\hat{x}^0, x^1, x^2, x^3), \quad \hat{W}_i = W_i(\hat{x}^0, x^1, x^2, x^3), \quad (5.3)$$

$$\hat{\beta}^{ij} = \beta^{ij}(\hat{x}^0, x^1, x^2, x^3), \quad \hat{P}^i = P^i(\hat{x}^0, x^1, x^2, x^3). \quad (5.4)$$

Applying the relationships (1.7) and (3.2) to (5.2), we get

$$\hat{g}_{ij} = g_{ij} + 2\varepsilon b_{ij} + \dots, \quad \hat{Q}_i = Q_i + \varepsilon W_i + \dots \quad (5.5)$$

In the case of (5.3) we use partial derivatives:

$$\hat{b}_{ij} = b_{ij} + \varepsilon \frac{\partial b_{ij}}{\partial x^0} + \dots, \quad \hat{W}_i = W_i + \varepsilon \frac{\partial W_i}{\partial x^0} + \dots \quad (5.6)$$

And in the case of (5.4) we we apply the relationships (4.1):

$$\begin{aligned} \hat{\beta}^{ij} &= \beta^{ij} + \varepsilon \frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} + \dots, \\ \hat{P}^i &= P^i + \varepsilon \frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} + \dots \end{aligned} \quad (5.7)$$

Through dots in (5.5), (5.6), (5.7), and in what follows below we denote higher order terms with respect to the small parameter  $\varepsilon \rightarrow 0$ .

The next step is to apply (5.1) to the integral (4.3) taking into account (4.2):

$$\begin{aligned} \hat{E}(\Omega) &= \int_{\Omega} \left( \sum_{i=1}^3 \sum_{j=1}^3 \beta^{ij} b_{ij} + \sum_{i=1}^n P^i W_i \right) \sqrt{\det \hat{g}} d^3x + \\ &+ \varepsilon \int_{\Omega} \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} b_{ij} \sqrt{\det g} d^3x + \\ &+ \varepsilon \int_{\Omega} \sum_{i=1}^3 \sum_{j=1}^3 \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} \frac{\partial b_{ij}}{\partial x^0} \sqrt{\det g} d^3x + \\ &+ \varepsilon \int_{\Omega} \sum_{i=1}^n \frac{\partial}{\partial x^0} \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} W_i \sqrt{\det g} d^3x + \\ &+ \varepsilon \int_{\Omega} \sum_{i=1}^n \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} \frac{\partial W_i}{\partial x^0} \sqrt{\det g} d^3x - \hat{L}(Q) + \dots \end{aligned} \quad (5.8)$$

The last term  $\hat{L}(Q)$  in (5.8) is determined by the Lagrangian  $\mathcal{L}$  in (4.2):

$$\hat{L}(Q) = \int_{\Omega} \hat{\mathcal{L}} \sqrt{\det \hat{g}} d^3x. \quad (5.9)$$

In order to transform (5.9) we should note that the formulas (5.5) and (5.6) are similar to small variations of the tensor fields  $\mathbf{g}$  and  $\mathbf{b}$  and to small variations of the dynamic variables of matter  $Q_1, \dots, Q_n$  and  $W_1, \dots, W_n$ :

$$\begin{aligned} \hat{g}_{ij} &= g_{ij} + \varepsilon h_{ij} + \dots, & \hat{Q}_i &= Q_i + \varepsilon h_i + \dots, \\ \hat{b}_{ij} &= b_{ij} + \varepsilon \eta_{ij} + \dots, & \hat{W}_i &= W_i + \varepsilon \eta_i + \dots \end{aligned} \quad (5.10)$$

The functions  $h_{ij}$ ,  $h_i$ ,  $\eta_{ij}$ , and  $\eta_i$  in (5.10) are functions with compact support (see [13]). They are applied to the integral over the whole 3D-brane universe:

$$L = \int \mathcal{L} \sqrt{\det g} d^3x. \quad (5.11)$$

Applying (5.10) to (5.11), we would write

$$\begin{aligned} \hat{L} = L + \varepsilon \int & \left( \sum_{i=1}^3 \sum_{j=1}^3 \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} \eta_{ij} + \sum_{i=1}^3 \sum_{j=1}^3 \left( \frac{\delta \mathcal{L}}{\delta g_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{b}} h_{ij} + \right. \\ & \left. + \sum_{i=1}^n \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} \eta_i + \sum_{i=1}^n \left( \frac{\delta \mathcal{L}}{\delta Q_i} \right)_{\mathbf{w}, \mathbf{g}, \mathbf{b}} h_i \right) \sqrt{\det g} d^3x + \dots \end{aligned} \quad (5.12)$$

The difference of (5.5) and (5.6) from (5.10) is that small variations in (5.5) and (5.6) are not functions with compact support. For this reason the analog of the formula (5.12) has an extra term with boundary integral:

$$\begin{aligned} \hat{L}(\Omega) = L(\Omega) + \varepsilon \int_{\Omega} & \sum_{i=1}^3 \sum_{j=1}^3 \left( \frac{\delta \mathcal{L}}{\delta b_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{g}} \frac{\partial b_{ij}}{\partial x^0} \sqrt{\det g} d^3x + \\ + \varepsilon \int_{\Omega} & \left( \sum_{i=1}^n \left( \frac{\delta \mathcal{L}}{\delta W_i} \right)_{\mathbf{Q}, \mathbf{g}, \mathbf{b}} \frac{\partial W_i}{\partial x^0} + \sum_{i=1}^n \left( \frac{\delta \mathcal{L}}{\delta Q_i} \right)_{\mathbf{w}, \mathbf{g}, \mathbf{b}} W_i \right) \sqrt{\det g} d^3x + \\ + \varepsilon \int_{\Omega} & \sum_{i=1}^3 \sum_{j=1}^3 \left( \frac{\delta \mathcal{L}}{\delta g_{ij}} \right)_{\mathbf{w}, \mathbf{Q}, \mathbf{b}} 2b_{ij} \sqrt{\det g} d^3x + \varepsilon \int_{\partial \Omega} (\mathcal{J}^1 dx^2 \wedge dx^3 + \\ & + \mathcal{J}^2 dx^3 \wedge dx^1 + \mathcal{J}^3 dx^1 \wedge dx^2) \frac{\sqrt{\det g}}{c} + \dots \end{aligned} \quad (5.13)$$

The extra term with boundary integral in (5.13) is associated with the energy flow. We shall study this term in the next section.

Let's return to the formula (5.8). The square root in the first integral of the formula (5.8) is transformed in the following way:

$$\sqrt{\det \hat{g}} = \sqrt{\det g} + \varepsilon \frac{\partial(\sqrt{\det g})}{\partial x^0} + \dots \quad (5.14)$$

This formula can be transformed further. It yields

$$\sqrt{\det \hat{g}} = \sqrt{\det g} + \varepsilon \sum_{q=1}^3 b_q^q \sqrt{\det g} + \dots \quad (5.15)$$

In deriving (5.15) from (5.14) we use the well-known Jacobi's formula for differentiating determinants (see [14]).

Now we can apply (5.13) and (5.15) to (5.8). In doing it we take into account the formulas (4.1) and the Euler-Lagrange equations (3.10) and (3.11). As a result the formula (5.8) reduces to the following one:

$$\begin{aligned} \hat{E}(\Omega) = E(\Omega) - \varepsilon \int_{\partial \Omega} & (\mathcal{J}^1 dx^2 \wedge dx^3 + \\ & + \mathcal{J}^2 dx^3 \wedge dx^1 + \mathcal{J}^3 dx^1 \wedge dx^2) \frac{\sqrt{\det g}}{c} + \dots \end{aligned} \quad (5.16)$$

On the other hand, applying the transformation (5.1) to the integral (4.3) directly, we obtain the following relationship:

$$\hat{E}(\Omega) = E(\Omega) + \varepsilon \frac{\partial E(\Omega)}{\partial x^0} + \dots \quad (5.17)$$

Comparing (5.16) and (5.17), we derive

$$\begin{aligned} \frac{\partial}{\partial x^0} \int_{\Omega} \mathcal{H} \sqrt{\det g} d^3x + \int_{\partial\Omega} (\mathcal{J}^1 dx^2 \wedge dx^3 + \\ + \mathcal{J}^2 dx^3 \wedge dx^1 + \mathcal{J}^3 dx^1 \wedge dx^2) \frac{\sqrt{\det g}}{c} = 0. \end{aligned} \quad (5.18)$$

The surface integral of the second kind in (5.18) can be transformed to a surface integral of the first kind. Indeed, we can write

$$\frac{\partial}{\partial x^0} \int_{\Omega} \mathcal{H} \sqrt{\det g} d^3x + \frac{1}{c} \int_{\partial\Omega} (\mathcal{J}^1 n_1 + \mathcal{J}^2 n_2 + \mathcal{J}^3 n_3) dS. \quad (5.19)$$

Here  $n_1, n_2, n_3$  are covariant components of the unit normal vector  $\mathbf{n}$  perpendicular to the boundary  $\partial\Omega$  and  $dS$  is the infinitesimal area element of the boundary. The quantities  $\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3$  in (5.19) are interpreted as the components a vector field. This vector field  $\mathbf{J}$  is interpreted as the density of the total energy flow. Since  $x^0 = ct$ , we can write the formula (5.19) as

$$\frac{\partial}{\partial t} \int_{\Omega} \mathcal{H} \sqrt{\det g} d^3x + \int_{\partial\Omega} \sum_{i=1}^3 \mathcal{J}^i n_i dS. \quad (5.20)$$

The equality (5.20) can be formulated as the following theorem.

**Theorem 5.1.** *The increment of the total energy of the gravitational field and matter per unit time in a closed 3D-domain  $\Omega$  is equal to the energy supplied to the domain per unit time through its boundary  $\partial\Omega$ .*

In order to transform the integral equality (5.20) to a differential form we apply the Ostrogradsky-Gauss formula (see [15]) along with the formulas (5.15) and (5.14). This yields the following differential relationship:

$$\frac{\partial \mathcal{H}}{\partial t} + \sum_{q=1}^3 c \mathcal{H} b_q^q + \sum_{i=1}^3 \nabla_i \mathcal{J}^i = 0. \quad (5.21)$$

The first term in (5.21) is the time derivative of the total energy density of the gravitational field and matter. The third term is the divergence of the density vector for the total energy flow. These two terms are standard. The second term in (5.21) is the Hubble term. It is associated with the Hubble expansion (see [16]) of a 3D-universe in our 3D-brane universe model. Note that some formulas including a Hubble term for the electromagnetic energy were derived in [17] within the standard four-dimensional paradigm.



## 6. DENSITY OF THE TOTAL ENERGY FLOW.

The vector  $\mathbf{J}$  with the components  $\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3$  arises in (5.13) when deriving an analog of the formula (5.12) where the small variations of the dynamic variables are not functions with compact support. We know that the Lagrangian (3.6) depends not only on the the functions in its argument list, but on their spatial derivatives of the form (3.7) and (3.8) as well. Let's choose  $b_{ij}[i_1 \dots i_s]$  in (3.8) and consider its entry to the Lagrangian (3.6). The variation of  $b_{ij}$  in (5.10) contributes to the variational expansion of the integral (5.9) through the term

$$I(\mathbf{b}) = \varepsilon \int_{\Omega} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_{ij}[i_1 \dots i_s] d^3x. \quad (6.1)$$

Let's denote through  $\iota_q$  a linear mapping acting upon differential 3-forms and producing differential 2-forms such that

$$\iota_q(dx^1 \wedge dx^2 \wedge dx^3) = \begin{cases} dx^2 \wedge dx^3 & \text{if } q = 1, \\ dx^3 \wedge dx^1 & \text{if } q = 2, \\ dx^1 \wedge dx^2 & \text{if } q = 3. \end{cases} \quad (6.2)$$

Now we can integrate (6.1) by parts. The result is written using (6.2):

$$\begin{aligned} & \varepsilon \int_{\Omega} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_{ij}[i_1 \dots i_s] d^3x = \\ & = \varepsilon \int_{\partial\Omega} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_{ij}[i_1 \dots i_{s-1}] \iota_{i_s}(dx^1 \wedge dx^2 \wedge dx^3) - \\ & \quad - \varepsilon \int_{\Omega} \frac{\partial}{\partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_{ij}[i_1 \dots i_{s-1}] d^3x. \end{aligned} \quad (6.3)$$

The last term in (6.3) is similar to the first term in it. Therefore we can iterate the integration by parts in (6.3). The result is written as follows:

$$\begin{aligned} I(\mathbf{b}) &= \sum_{r=1}^s \varepsilon \int_{\partial\Omega} (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \times \\ & \quad \times \eta_{ij}[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\ & \quad + \varepsilon \int_{\Omega} (-1)^s \frac{\partial^s}{\partial x^{i_1} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_{ij} d^3x. \end{aligned} \quad (6.4)$$

The last term in (6.4) contributes to the bulk integrals in (5.13). The previous terms contribute to the boundary integral in (5.13).

The variation of  $g_{ij}$  in (5.10) contributes to the variational expansion of the integral (5.9) through the following term:

$$I(\mathbf{g}) = \varepsilon \int_{\Omega} \left( \frac{\partial \mathcal{L}}{\partial g_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) h_{ij}[i_1 \dots i_s] d^3x. \quad (6.5)$$

Integrating by parts iteratively in (6.5), we derive a formula similar to (6.4):

$$\begin{aligned}
I(\mathbf{g}) &= \sum_{r=1}^s \varepsilon \int_{\partial\Omega} (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial g_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) \times \\
&\quad \times h_{ij}[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\
&\quad + \varepsilon \int_{\Omega} (-1)^s \frac{\partial^s}{\partial x^{i_1} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial g_{ij}[i_1 \dots i_s]} \sqrt{\det g} \right) h_{ij} d^3x.
\end{aligned} \tag{6.6}$$

Further two steps are similar to the previous two. The analogs of the formulas (6.1) and (6.5) for the dynamic variables of matter in (5.10) are

$$\begin{aligned}
I(\mathbf{W}) &= \varepsilon \int_{\Omega} \left( \frac{\partial \mathcal{L}}{\partial W_i[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_i[i_1 \dots i_s] d^3x, \\
I(\mathbf{Q}) &= \varepsilon \int_{\Omega} \left( \frac{\partial \mathcal{L}}{\partial Q_i[i_1 \dots i_s]} \sqrt{\det g} \right) h_i[i_1 \dots i_s] d^3x.
\end{aligned} \tag{6.7}$$

Integrating by parts iteratively in (6.7), we get formulas similar to (6.4) and (6.6):

$$\begin{aligned}
I(\mathbf{W}) &= \sum_{r=1}^s \varepsilon \int_{\partial\Omega} (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial W_i[i_1 \dots i_s]} \sqrt{\det g} \right) \times \\
&\quad \times \eta_i[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\
&\quad + \varepsilon \int_{\Omega} (-1)^s \frac{\partial^s}{\partial x^{i_1} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial W_i[i_1 \dots i_s]} \sqrt{\det g} \right) \eta_i d^3x, \\
I(\mathbf{Q}) &= \sum_{r=1}^s \varepsilon \int_{\partial\Omega} (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial Q_i[i_1 \dots i_s]} \sqrt{\det g} \right) \times \\
&\quad \times h_i[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\
&\quad + \varepsilon \int_{\Omega} (-1)^s \frac{\partial^s}{\partial x^{i_1} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial Q_i[i_1 \dots i_s]} \sqrt{\det g} \right) h_i d^3x.
\end{aligned} \tag{6.8}$$

The quantities  $\eta_{ij}$ ,  $h_{ij}$ ,  $\eta_i$ , and  $h_i$  in the formulas (6.4), (6.6), (6.8), and (6.9) should be replaced with the following ones:

$$\eta_{ij} = \frac{\partial b_{ij}}{\partial x^0}, \quad \eta_i = \frac{\partial W_i}{\partial x^0}, \tag{6.10}$$

$$h_{ij} = 2 b_{ij}, \quad h_i = W_i, \tag{6.11}$$

The formulas (6.10) and (6.11) are derived by comparing (5.5) and (5.6) with (5.10).

The last step in calculating the components of the vector  $\mathbf{J}$  consists in collecting boundary terms from (6.4), (6.6), (6.8), and (6.9) into one formula. Assume  $N$  to be the maximal order of the partial derivatives of the form (3.7) and (3.8) in  $\mathcal{L}$ . Then from (6.4), (6.6), (6.8), (6.9) and (5.13) we derive

$$\begin{aligned}
& (\mathcal{J}^1 dx^2 \wedge dx^3 + \mathcal{J}^2 dx^3 \wedge dx^1 + \mathcal{J}^3 dx^1 \wedge dx^2) \frac{\sqrt{\det g}}{c} = \\
& = \sum_{i=1}^3 \sum_{j=1}^3 \sum_{s=1}^N \sum_{r=1}^s (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \dots i_s]} \times \right. \\
& \quad \left. \times \sqrt{\det g} \right) \eta_{ij}[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\
& + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{s=1}^N \sum_{r=1}^s (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial g_{ij}[i_1 \dots i_s]} \times \right. \\
& \quad \left. \times \sqrt{\det g} \right) h_{ij}[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\
& + \sum_{i=1}^n \sum_{s=1}^N \sum_{r=1}^s (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial W_i[i_1 \dots i_s]} \times \right. \\
& \quad \left. \times \sqrt{\det g} \right) \eta_i[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\
& + \sum_{i=1}^n \sum_{s=1}^N \sum_{r=1}^s (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}} \left( \frac{\partial \mathcal{L}}{\partial Q_i[i_1 \dots i_s]} \times \right. \\
& \quad \left. \times \sqrt{\det g} \right) h_i[i_1 \dots i_{s-r}] \iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3).
\end{aligned} \tag{6.12}$$

Note that the formulas (6.10) and (6.11) apply to (6.12) as well as to the previous formulas (6.1), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), and (6.9). Note also that the partial differentiation operators of the form

$$\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \dots \partial x^{i_s}}$$

are omitted in those terms of (6.12) where  $r = 1$ . The same is true for all previous formulas where these operators are used.

## 7. CONCLUDING REMARKS.

The main result of the present paper is Theorem 5.1 presenting the total energy conservation law for the gravitational field and matter in a 3D-brane universe. This law is expressed by the formula (5.19) or equivalently by the formula (5.20). The components  $\mathcal{J}^1$ ,  $\mathcal{J}^2$ ,  $\mathcal{J}^3$  of the vector  $\mathbf{J}$  expressing the density of the total energy flow are given by the formula (6.12), where  $\mathcal{L}$  is the total Lagrangian (3.5) of the gravitational field and matter.

In the standard four-dimensional paradigm the energy conservation law is united with the momentum conservation law and is formulated as the energy-momentum conservation law which is expressed by the formula

$$\sum_{q=0}^3 \nabla_q T^{kq} = 0, \tag{7.1}$$

see (3.6) in § 3 of Chapter V of [6]. Our formula (5.21) expressing the total energy conservation law in differential form replaces in part the formula (7.1). As for the momentum conservation law, in the 3D-brane universe paradigm it is a separate law that should be studied separately. This will be done in a separate paper.

## 8. DEDICATORY.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

## REFERENCES

1. Sharipov R. A., *A three-dimensional brane universe in a four-dimensional spacetime with a Big Bang*, e-print [viXra:2207.0173](https://arxiv.org/abs/2207.0173).
2. Sharipov R. A., *On the dynamics of a 3D universe in a 4D spacetime*, Conference abstracts book “Ufa autumn mathematical school 2022” (Fazullin Z. Yu., ed.), vol. 2, pp. 279–281; DOI: 10.33184/mnkuomsh2t-2022-09-28.104.
3. Sharipov R. A., *The universe as a 3D brane and the equations for it*, Conference abstracts book “Fundamental mathematics and its applications in natural sciences 2022” (Gabdrakhmanova L. A., ed.), p. 37; DOI: 10.33184/fmpve2022-2022-10-19.30.
4. *Cosmic time*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
5. *Comoving and proper distances*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
6. Sharipov R. A., *Classical electrodynamics and theory of relativity*, Bashkir State University, Ufa, 1997; see also [arXiv:physics/0311011](https://arxiv.org/abs/physics/0311011).
7. *Gravitational constant*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
8. *Cosmological constant*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
9. *Stress-energy tensor*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
10. Sharipov R. A., *Course of differential geometry*, Bashkir State University, Ufa, 1996; see also [arXiv:math/0412421](https://arxiv.org/abs/math/0412421).
11. Sharipov R. A., *Lagrangian approach to deriving the gravity equations for a 3D-brane universe*, e-print [viXra:2301.0033](https://arxiv.org/abs/2301.0033).
12. Sharipov R. A., *Hamiltonian approach to deriving the gravity equations for a 3D-brane universe*, e-print [viXra:2302.0120](https://arxiv.org/abs/2302.0120).
13. *Compact support*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
14. *Jacobi's formula*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
15. *Divergence theorem*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
16. *Hubble's law*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
17. Sharipov R. A., *A note on electromagnetic energy in the context of cosmology*, e-print [viXra:2207.0092](https://arxiv.org/abs/2207.0092).

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