ENERGY CONSERVATION LAW FOR THE GRAVITATIONAL FIELD IN A 3D-BRANE UNIVERSE WITHOUT EQUIDISTANCE POSTULATE.

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Abstract. The 3D-brane universe model is an alternative non-Einsteinian theory of gravity. The initial version of this theory was built using the so-called equidistance postulate. Its second version is free of the equidistance postulate. In the present paper we rederive the energy conservation law for the gravitational field within the framework of the second extended version of the theory.

1. Introduction.

The initial version of the 3D-brane universe model was built in the series of several papers [1–6] (see also [7–11]). Its second version was started in [12]. In both versions of the theory the four-dimensional spacetime is assumed to be foliated into 3D-branes. This foliation is a new geometric structure absent in the standard relativity. All 3D-branes of this structure are space-like three-dimensional hypersurfaces. Their unit normal vectors are time-like vectors. They constitute a vector field. We denote it through $\mathbf{n}$.

Definition 1.1. An observer whose world line (see [13]) is perpendicular to all 3D-branes in the foliation of the 3D-branes in the four-dimensional spacetime is called a comoving observer.

It is easy to see that world lines of comoving observers in the spacetime do coincide with integral curves of the vector field of unit vectors $\mathbf{n}$.

Remark. *Vector fields of time-like unit vectors are considered in Einstein aether theory (see* [14]*). However vector fields considered there are not produced by foliations of 3D-hypersurfaces and in general case they do not produce hypersurfaces perpendicular to them.*

The initial version of the 3D-brane universe model was built using the equidistance postulate. This postulate is formulated as follows.

Postulate 1.1. *Watches of any two comoving observers can be synchronized.*

Having synchronized the watches of all comoving observers, one can use their common time as a global cosmological time. This was done in the initial version of the theory in [1–6] (see also [7–11]). If a universe in question originates from a

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single singular point, which is called the Big Bang, then this point can be used as a starting reference point for the global cosmological time. In this case the global cosmological time becomes unique.

In the new version of the 3D-brane universe model the postulate 1.1 is excluded. In this version of the theory there is no global cosmological time. In [12] it was replaced by the so-called brane time. Formally it can be defined as follows.

Definition 1.2. Any smooth function t in the four-dimensional spacetime with nonvanishing time-like gradient directed to the future can be chosen for a brane time if it is constant throughout each 3D-brane of the foliation of 3D-branes in the spacetime and if its values are measured in time units.

Unlike the global cosmological time in the initial version of the theory, the choice of the brane time is not unique. Any two brane time variables t and $\tilde{t}$ are related to each other through the following formulas:

$$\tilde{t} = \tilde{t}(t), \qquad\qquad t = t(\tilde{t}). \tag{1.1}$$

Here $\tilde{t}(t)$ and $t(\tilde{t})$ are two mutually inverse strictly growing smooth functions of one variable. The transformations (1.1) are called time scaling transformations.

Definition 1.3. Any smooth function x in the four-dimensional spacetime with nonvanishing space-like gradient is called a comoving function if it is constant along world lines of all comoving observers.

Definition 1.4. Any three comoving functions x^1, x^2, x^3 in the four-dimensional spacetime whose gradients are linearly independent constitute comoving spacial coordinates in the spacetime.

Comoving spacial coordinates x^1, x^2, x^3 are usually complemented with a temporal coordinate x^0. In the 3D-brane universe model it can be done as follows:

$$x^0 = c_{\text{gr}}\, t. \tag{1.2}$$

Here t is a brane time and c_{gr} is a speed constant which is interpreted as the speed of gravitational waves. In the standard relativity this constants coincides with the speed light c_{gr} (speed of electromagnetic waves). In our theory these two constants a priory can be different (see [5]).

The temporal coordinate (1.2) was used in previous papers [1–4] and [12]. However, below in the present paper we shall not use it in order to increase the extent of three-dimensionality in the theory. Instead of the temporal coordinate (1.2) we shall use the time variable t itself. Therefore in the second version of the 3D-brane universe model, which is the newest at the moment, the gravitational field is described by the time-dependent three-dimensional metric

$$g_{ij} = g_{ij}(t, x^1, x^2, x^3), \quad 1 \leqslant i, j \leqslant 3, \tag{1.3}$$

and by the time-dependent scalar function

$$g_{00} = g_{00}(t, x^1, x^2, x^3). \tag{1.4}$$

The equations for the metric (1.3) and the function (1.4) were derived in [12]. Then they were rederived using the Lagrangian approach in [15]. In the present paper we use the Lagrangian approach from [15] and the technique from [4] in order to derive the energy conservation law within the second version of the 3D-brane universe model in the absence of the equidistance postulate 1.1.

2. Action integrals for the gravitational field and matter.

Action integrals in field theories are usually written as time integrals of Lagrangians, while Lagrangians are spacial integrals of Lagrangian densities. Therefore we write the action integral for the gravitational field as follows:

$$S_{\text{gr}} = \int L_{\text{gr}}\,dt, \qquad\qquad L_{\text{gr}} = \int \mathcal{L}_{\text{gr}}\,\sqrt{\det g}\;d^3x. \tag{2.1}$$

Matter has its own action integral and its own Lagrangian:

$$S_{\text{mat}} = \int L_{\text{mat}}\,dt, \qquad\qquad L_{\text{mat}} = \int \mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x. \tag{2.2}$$

The Lagrangian density for the gravitational field $\mathcal{L}_{\text{gr}}$ in (2.1) is taken from [15]. It is given by the following formula:

$$\mathcal{L}_{\text{gr}} = -\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\sqrt{g_{00}}\,(\rho + 2\,\Lambda). \tag{2.3}$$

The scalar function ρ in (2.3) is an auxiliary notation:

$$\begin{aligned} \rho = g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\,g_{00} - \frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,-\\ -R - g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b_q^k\,b_k^q + g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b_k^k\,b_q^q. \end{aligned} \tag{2.4}$$

The constant c_{gr} in (2.3) is the same speed constant as in (1.2), γ is Newton's gravitational constant (see [16]), and Λ is the cosmological constant (see [17]). The quantities b_j^i (2.4) are components of a three-dimensional time-dependent tensor field. They are given by the following formulas:

$$b_{ij} = \frac{\dot g_{ij}}{2\,c_{\text{gr}}} = \frac{1}{2\,c_{\text{gr}}}\,\frac{\partial g_{ij}}{\partial t}, \qquad\qquad b_j^i = \sum_{q=1}^{3} g^{iq}\,b_{qj}. \tag{2.5}$$

The second formula in (2.5) is the standard index raising procedure by means of the inverse metric tensor (see § 9 of Chapter II in [18]).

Due to (2.4) the Lagrangian density (2.3) depends on g_{00} from (1.4), on g_{ij} from (1.3) and on the time derivatives of these dynamic variables. The time derivatives of g_{ij} are expressed through b_{ij} using (2.5). By analogy to (2.5) we write

$$b_{00} = \frac{\dot g_{00}}{c_{\text{gr}}} = \frac{1}{c_{\text{gr}}}\,\frac{\partial g_{00}}{\partial t}, \qquad\qquad b_0^0 = g_{00}^{-1}\,b_{00}. \tag{2.6}$$

Using (2.6), one can express the time derivative of g_{00} through b_{00}. Therefore we write the following formal expression for the Lagrangian density (2.3):

$$\mathcal{L}_{\mathrm{gr}} = \mathcal{L}_{\mathrm{gr}}(g, b, \mathbf{g}, \mathbf{b}). \tag{2.7}$$

Here g and b represent g_{00} and b_{00}, while $\mathbf{g}$ and $\mathbf{b}$ represent g_{ij} and b_{ij}. The Lagrangian density of matter can depend on some auxiliary dynamic variables responsible for the state of matter. We denote these auxiliary dynamic variables through $Q_1, \ldots, Q_n$ and their time derivatives through $W_1, \ldots, W_n$:

$$W_i = \dot{Q}_i = \frac{\partial Q_i}{\partial t}. \tag{2.8}$$

The formula (2.8) is an analogue of the formulas (2.5) and (2.6). Note that it is slightly different from the similar formula (3.2) in [4].

The quantities $Q_1, \ldots, Q_n$ and $W_1, \ldots, W_n$ are fields, i. e. they depend on the spatial variables x^1, x^2, x^3 and on the time variable t:

$$Q_i = Q_i(t, x^1, x^2, x^3), \qquad\qquad W_i = W_i(t, x^1, x^2, x^3). \tag{2.9}$$

The same is true for the components g_{ij} and b_{ij} of the fields $\mathbf{g}$ and $\mathbf{b}$ and for the scalar fields g_{00} and b_{00}.

In this paper we do not consider any specific sort of matter. Therefore we do not write explicit formulas for the Lagrangian density of matter $\mathcal{L}_{\mathrm{mat}}$ and use only a formal expression for it similar to that of (2.7):

$$\mathcal{L}_{\mathrm{mat}} = \mathcal{L}_{\mathrm{mat}}(\mathbf{Q}, \mathbf{W}, g, b, \mathbf{g}, \mathbf{b}). \tag{2.10}$$

Through $\mathbf{Q}$ and $\mathbf{W}$ in (2.10) we denote the lists of quantities $Q_1, \ldots, Q_n$ and $W_1, \ldots, W_n$ presented in (2.9).

The total action integral S for the gravitational field and matter is the sum of action integrals S_{gr} and S_{mat} from (2.1) and (2.2):

$$S = S_{\mathrm{gr}} + S_{\mathrm{mat}}. \tag{2.11}$$

The same is true for the Lagrangians and for their densities:

$$L = L_{\mathrm{gr}} + L_{\mathrm{mat}}, \qquad\qquad \mathcal{L} = \mathcal{L}_{\mathrm{gr}} + \mathcal{L}_{\mathrm{mat}}. \tag{2.12}$$

The formulas (2.12) are immediate from (2.11), (2.1), and (2.2). Applying (2.7) and (2.10) to the second formula (2.12), we derive

$$\mathcal{L} = \mathcal{L}(\mathbf{Q}, \mathbf{W}, g, b, \mathbf{g}, \mathbf{b}). \tag{2.13}$$

Then we can write a formula similar to (2.1) and (2.2):

$$S = \int L\,dt, \qquad\qquad L = \int \mathcal{L}\,\sqrt{\det g}\;d^3x. \tag{2.14}$$

The formula (2.13) is a formal expression for the total Lagrangian density $\mathcal{L}$ in (2.14). It is similar to (2.7) and (2.10).

3. Euler-Lagrange equations.

The next step is to apply the stationary action principle (see [19]) to the total action integral in (2.14). As a result we get a series of Euler-Lagrange equations. This series subdivides into three groups according to the groups of dynamic variables. The first group of Euler-Lagrange equations is associated with the metric components g_{ij} and their time derivatives in (2.5):

$$-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}-\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}=0. \tag{3.1}$$

The second group of equations is associated with the scalar field g_{00} and its time derivative in (2.6). This group comprises only one equation:

$$-\frac{1}{c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}-\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{00}}\Bigr)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}=0. \tag{3.2}$$

The third group of equations is associated with dynamic variables $Q_1,\,\ldots\,,Q_n$ of matter and their time derivatives $W_1,\,\ldots\,,W_n$ in (2.8):

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}-c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{W}}}=0. \tag{3.3}$$

More details concerning the equations (3.1), (3.2), and (3.3) can be found in [15].

4. Legendre transformation and the density of energy.

The quantities g_{ij}, g_{00}, and Q_i are treated as generalized coordinates, while the quantities b_{ij}, b_{00}, and W_i are corresponding generalized velocities. The Legendre transformation (see [20]) maps generalized velocities to generalized momenta. In our present case it is given by the following formulas:

$$\beta^{ij}=\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}},\qquad \beta^{00}=\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}},\qquad P^i=\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}. \tag{4.1}$$

Then, using (4.1), the total energy density is defined by the formula

$$\mathcal{H}=\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\beta^{00}\,b_{00}+\sum_{i=1}^{n}P^i\,W_i-\mathcal{L}. \tag{4.2}$$

Let Ω be a three-dimensional domain in a 3D-brane universe. The energy of the gravitational field and matter within this domain is given by the following integral:

$$E(\Omega)=\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x. \tag{4.3}$$

The main goal of the next section is to derive a formula for the time derivative of the energy integral (4.3).

5. The energy conservation law.

Let's consider a small increment of the time variable t. We write it as follows:

$$\hat{t}=t+\varepsilon. \tag{5.1}$$

Then we apply (5.1) to the dynamic variables of the gravitational field and matter:

$$\hat{g}_{ij}=g_{ij}(\hat{t},x^1,x^2,x^3), \qquad \hat{Q}_i=Q_i(\hat{t},x^1,x^2,x^3), \tag{5.2}$$

$$\hat{b}_{ij}=b_{ij}(\hat{t},x^1,x^2,x^3), \qquad \hat{W}_i=W_i(\hat{t},x^1,x^2,x^3), \tag{5.3}$$

$$\hat{\beta}^{ij}=\beta^{ij}(\hat{t},x^1,x^2,x^3), \qquad \hat{P}^i=P^i(\hat{t},x^1,x^2,x^3), \tag{5.4}$$

$$\hat{g}_{00}=g_{00}(\hat{t},x^1,x^2,x^3), \qquad \hat{b}_{00}=b_{00}(\hat{t},x^1,x^2,x^3), \tag{5.5}$$

$$\hat{\beta}^{00}=\beta^{00}(\hat{t},x^1,x^2,x^3). \tag{5.6}$$

Applying the relationships (2.5) and (2.8) to (5.2), we get

$$\hat{g}_{ij}=g_{ij}+2\,c_{\text{gr}}\,\varepsilon\,b_{ij}+\ldots, \qquad \hat{Q}_i=Q_i+\varepsilon\,W_i+\ldots. \tag{5.7}$$

In the case of (5.3) we use partial derivatives:

$$\hat{b}_{ij}=b_{ij}+\varepsilon\,\frac{\partial b_{ij}}{\partial t}+\ldots, \qquad \hat{W}_i=W_i+\varepsilon\,\frac{\partial W_i}{\partial t}+\ldots. \tag{5.8}$$

And in the case of (5.4) and (5.6) we we apply the relationships (4.1):

$$\begin{aligned}
&\hat{\beta}^{ij}=\beta^{ij}+\varepsilon\,\frac{\partial}{\partial t}\Big(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Big)_{\substack{g,b,\mathbf{g},\\ \mathbf{Q},\mathbf{W}}}+\ldots,\\
&\hat{P}^i=P^i+\varepsilon\,\frac{\partial}{\partial t}\Big(\frac{\delta\mathcal{L}}{\delta W_i}\Big)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}+\ldots,\\
&\hat{\beta}^{00}=\beta^{00}+\varepsilon\,\frac{\partial}{\partial t}\Big(\frac{\delta\mathcal{L}}{\delta b_{00}}\Big)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}+\ldots.
\end{aligned} \tag{5.9}$$

In the case of (5.5) we apply (2.6) and use the partial derivative of b_{00}:

$$\hat{g}_{00}=g_{00}+c_{\text{gr}}\,\varepsilon\,b_{00}+\ldots, \qquad \hat{b}_{00}=b_{00}+\varepsilon\,\frac{\partial b_{00}}{\partial t}+\ldots. \tag{5.10}$$

Through dots in (5.7), (5.8), (5.9), (5.10), and in what follows below we denote higher order terms with respect to the small parameter $\varepsilon\to 0$.

In addition to (5.7), (5.8), (5.9), (5.10) we consider the following relationship:

$$\sqrt{\det\hat{g}}\,=\sqrt{\det g}\,\Big(1+\varepsilon\,c_{\text{gr}}\sum_{k=1}^{3}b^k_k+\ldots\Big). \tag{5.11}$$

The relationship (5.11) is derived using the well-known Jacobi formula for differentiating determinants (see [21]) and the relationships (2.5). Like in the previous formulas, through dots in (5.11) we denote higher order terms with respect to $\varepsilon\to 0$.

The next step is to apply (5.1) to the integral (4.3) taking into account (4.2):

$$\begin{aligned}
\hat{E}(\Omega)=\int\limits_{\Omega}\biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\beta^{00}\,b_{00}+\sum_{i=1}^{n}P^{i}\,W_i\biggr)\sqrt{\det\hat{g}}\;d^3x\,+\\
+\,\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\biggl(\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}b_{ij}+\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\frac{\partial b_{ij}}{\partial t}\biggr)\sqrt{\det g}\;d^3x\,+\\
+\,\varepsilon\int\limits_{\Omega}\sum_{i=1}^{n}\biggl(\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}W_i+\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\frac{\partial W_i}{\partial t}\biggr)\sqrt{\det g}\;d^3x\,+\\
+\,\varepsilon\int\limits_{\Omega}\biggl(\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}b_{00}+\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\frac{\partial b_{00}}{\partial t}\biggr)\sqrt{\det g}\;d^3x-\hat{L}(\Omega)+\ldots.
\end{aligned}\tag{5.12}$$

The last term $\hat{L}(\Omega)$ in (5.12) is determined by the Lagrangian density $\mathcal{L}$ in (4.2):

$$\hat{L}(\Omega)=\int\limits_{\Omega}\hat{\mathcal{L}}\sqrt{\det\hat{g}}\;d^3x.\tag{5.13}$$

In order to transform (5.13) we should note that the formulas (5.7), (5.8), and (5.10) are similar to small variations of the tensor fields $\mathbf{g}$ and $\mathbf{b}$, to small variations of the dynamic variables of matter $Q_1,\,\ldots\,,\,Q_n$ and $W_1,\,\ldots\,,\,W_n$, and to small variations of the scalar fields g_{00} and b_{00} in calculus of variations:

$$\begin{aligned}
&\hat{g}_{ij}=g_{ij}+\varepsilon\,h_{ij}+\ldots\,, &&\hat{Q}_i=Q_i+\varepsilon\,h_i+\ldots\,,\\
&\hat{b}_{ij}=b_{ij}+\varepsilon\,\eta_{ij}+\ldots\,, &&\hat{W}_i=W_i+\varepsilon\,\eta_i+\ldots\,,\\
&\hat{g}_{00}=g_{00}+\varepsilon\,h_{00}+\ldots\,, &&\hat{b}_{00}=b_{00}+\varepsilon\,\eta_{00}+\ldots\,.
\end{aligned}\tag{5.14}$$

The functions h_{ij}, h_i, η_{ij}, η_i, h_{00}, and η_{00} in (5.14) are functions with compact support (see [22]). They are applied to the integral over the whole 3D-brane universe:

$$L=\int\mathcal{L}\sqrt{\det g}\;d^3x.\tag{5.15}$$

Applying (5.14) to (5.15), we would write

$$\begin{aligned}
\hat{L}=L+\varepsilon\int\biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\eta_{ij}+\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}h_{ij}\,+\\
+\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\eta_{00}+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{00}}\Bigr)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}h_{00}\,+\\
+\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\eta_i+\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{W}}}h_i\biggr)\sqrt{\det g}\;d^3x\,+\ldots.
\end{aligned}\tag{5.16}$$

The difference of (5.7), (5.8), and (5.10) from (5.14) is that small variations in (5.7), (5.8), and (5.10) are not functions with compact support. For this reason the analog of the formula (5.16) has an extra term with a boundary integral:

$$
\begin{aligned}
\hat{L}(\Omega) &= L(\Omega) + \varepsilon \int\limits_{\Omega} \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}} \frac{\partial b_{ij}}{\partial t}\,\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\Biggl(\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\frac{\partial W_i}{\partial t}+\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{W}}} W_i\Biggr)\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\Biggl(\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\frac{\partial b_{00}}{\partial t}+\Bigl(\frac{\delta\mathcal{L}}{\delta g_{00}}\Bigr)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}} c_{\text{gr}}\,b_{00}\Biggr)\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}} 2\,c_{\text{gr}}\,b_{ij}\,\sqrt{\det g}\;d^3x+\varepsilon\int\limits_{\partial\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3\,+\\
&\qquad+\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\sqrt{\det g}+\ldots\,.
\end{aligned}
\tag{5.17}
$$

The extra term with the boundary integral in (5.17) is associated with the energy flow. We shall study this term in the next section.

Let's return to the formula (5.12). The square root in the first integral of the formula (5.8) is transformed with the use of the formula (5.11). Now we can apply (5.11) and (5.17) to (5.12). In doing it we take into account the formulas (4.1) and the Euler-Lagrange equations (3.1), (3.2), and (3.3). As a result the formula (5.12) reduces to the following one:

$$
\begin{aligned}
\hat{E}(\Omega) = E(\Omega) - \varepsilon \int\limits_{\partial\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3\,+&\\
+\,\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\sqrt{\det g}+\ldots\,.&
\end{aligned}
\tag{5.18}
$$

On the other hand, applying the transformation (5.1) to the integral (4.3) directly, we obtain the following relationship:

$$
\hat{E}(\Omega) = E(\Omega) + \varepsilon\,\frac{\partial E(\Omega)}{\partial t}+\ldots\,.
\tag{5.19}
$$

Comparing (5.18) and (5.19), we derive

$$
\begin{aligned}
\frac{\partial}{\partial t}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3\,+&\\
+\,\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\sqrt{\det g}=0.&
\end{aligned}
\tag{5.20}
$$

The surface integral of the second kind in (5.20) can be transformed to a surface integral of the first kind. Indeed, we can write

$$
\frac{\partial}{\partial t}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\Omega}\bigl(\mathcal{J}^1\,n_1+\mathcal{J}^2\,n_2+\mathcal{J}^3\,n_3\bigr)\,dS.
\tag{5.21}
$$

Here n_1, n_2, n_3 are covariant components of the unit normal vector $\mathbf{n}$ perpendicular to the boundary $\partial\,\Omega$ and dS is the infinitesimal area element of the boundary. The quantities $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ in (5.21) are interpreted as the components a vector field. This vector field $\mathbf{J}$ is interpreted as the density of the total energy flow:

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\,d^3x+\int\limits_{\partial\,\Omega}\sum_{i=1}^{3}\mathcal{J}^i\,n_i\,dS. \tag{5.22}$$

The equality (5.22) can be formulated as the following theorem.

Theorem 5.1. *The increment of the total energy of the gravitational field and matter per unit time in a closed 3D-domain Ω is equal to the energy supplied to the domain per unit time through its boundary $\partial\,\Omega$.*

In order to transform the integral equality (5.22) to a differential form we apply the Ostrogradsky-Gauss formula (see [23]) along with the formula (5.11). This yields the following differential relationship:

$$\frac{\partial\mathcal{H}}{\partial t}+\sum_{q=1}^{3}c_{\text{gr}}\,\mathcal{H}\,b^q_q+\sum_{i=1}^{3}\nabla_i\mathcal{J}^i=0. \tag{5.23}$$

The first term in (5.21) is the time derivative of the total energy density of the gravitational field and matter. The third term is the divergence of the density vector for the total energy flow. These two terms are standard. The second term in (5.23) is the Hubble term. It is associated with the Hubble expansion (see [24]) of a 3D-universe in our 3D-brane universe model. Note that some formulas including a Hubble term for the electromagnetic energy were derived in [25] within the standard four-dimensional paradigm.

6. Density of the total energy flow.

The vector $\mathbf{J}$ with the components $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ arises in (5.17) when deriving an analog of the formula (5.16) where the small variations of the dynamic variables are not functions with compact support. We know that the Lagrangian (2.13) depends not only on the the functions in its argument list, but on some finite number of its partial derivatives with respect to the spatial variables x^1, x^2, x^3. For this reason we introduce the following notations similar to those from [4]:

$$Q_i[i_1\ldots i_s]=\frac{\partial Q_i}{\partial x^{i_1}\ldots\partial x^{i_s}},\qquad W_i[i_1\ldots i_s]=\frac{\partial W_i}{\partial x^{i_1}\ldots\partial x^{i_s}}. \tag{6.1}$$

$$g_{ij}[i_1\ldots i_s]=\frac{\partial g_{ij}}{\partial x^{i_1}\ldots\partial x^{i_s}},\qquad b_{ij}[i_1\ldots i_s]=\frac{\partial b_{ij}}{\partial x^{i_1}\ldots\partial x^{i_s}}. \tag{6.2}$$

$$g_{00}[i_1\ldots i_s]=\frac{\partial g_{00}}{\partial x^{i_1}\ldots\partial x^{i_s}},\qquad b_{00}[i_1\ldots i_s]=\frac{\partial b_{00}}{\partial x^{i_1}\ldots\partial x^{i_s}}. \tag{6.3}$$

Let's choose the quantity $b_{ij}[i_1\ldots i_s]$ in (6.2) and consider its entry to the Lagrangian (2.13). The variation of b_{ij} in (5.14) contributes to the variational expansion of the integral (5.13) through the term

$$I(\mathbf{b})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\ldots i_s]\,d^3x. \tag{6.4}$$

Let's denote through ι_q a linear mapping acting upon differential 3-forms and producing differential 2-forms such that

$$\iota_q(dx^1\wedge dx^2\wedge dx^3)=\begin{cases} dx^2\wedge dx^3 & \text{if}\ \ q=1,\\ dx^3\wedge dx^1 & \text{if}\ \ q=2,\\ dx^1\wedge dx^2 & \text{if}\ \ q=3.\end{cases}\tag{6.5}$$

Now we can integrate (6.4) by parts. The result is written using (6.5):

$$\begin{aligned}\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_s]\,d^3x=\\ =\varepsilon\int\limits_{\partial\,\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_{s-1}]\,\iota_{i_s}(dx^1\wedge dx^2\wedge dx^3)\,-\\ -\,\varepsilon\int\limits_{\Omega}\frac{\partial}{\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_{s-1}]\,d^3x.\end{aligned}\tag{6.6}$$

The last term in (6.6) is similar to the first term in it. Therefore we can iterate the integration by parts in (6.6). The result is written as follows:

$$\begin{aligned}I(\mathbf{b})=\sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\times\\ \times\,\eta_{ij}[i_1\dots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}\,d^3x.\end{aligned}\tag{6.7}$$

The last term in (6.7) contributes to the bulk integrals in (5.17). The previous terms contribute to the boundary integral in (5.17).

The variation of g_{ij} in (6.2) contributes to the variational expansion of the integral (5.13) through the following term:

$$I(\mathbf{g})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{ij}[i_1\dots i_s]\,d^3x.\tag{6.8}$$

Integrating by parts iteratively in (6.8), we derive a formula similar to (6.7):

$$\begin{aligned}I(\mathbf{g})=\sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\times\\ \times\,h_{ij}[i_1\dots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{ij}\,d^3x.\end{aligned}\tag{6.9}$$

Further two steps are similar to the previous two. The analogs of the formulas (6.4) and (6.8) for the dynamic variables of matter in (6.1) are

$$\begin{aligned} I(\mathbf{W}) &= \varepsilon\int\limits_{\Omega}\Big(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,\eta_i[i_1\ldots i_s]\,d^3x,\\ I(\mathbf{Q}) &= \varepsilon\int\limits_{\Omega}\Big(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,h_i[i_1\ldots i_s]\,d^3x. \end{aligned}\tag{6.10}$$

Integrating by parts iteratively in (6.10), we get formulas similar to (6.7) and (6.9):

$$\begin{aligned} I(\mathbf{W}) = \sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Big(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\times\\ \times\,\eta_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Big(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,\eta_i\,d^3x, \end{aligned}\tag{6.11}$$

$$\begin{aligned} I(\mathbf{Q}) = \sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Big(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\times\\ \times\,h_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Big(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,h_i\,d^3x. \end{aligned}\tag{6.12}$$

Then we proceed to the variations of the scalar fields g_{00} and b_{00} in (6.3). The analogs of the integrals from (6.10) in this case are written as

$$\begin{aligned} I(b) &= \varepsilon\int\limits_{\Omega}\Big(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,\eta_{00}[i_1\ldots i_s]\,d^3x,\\ I(g) &= \varepsilon\int\limits_{\Omega}\Big(\frac{\partial\mathcal{L}}{\partial g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,h_{00}[i_1\ldots i_s]\,d^3x. \end{aligned}\tag{6.13}$$

Integrating by parts iteratively the first integral (6.13), we get

$$\begin{aligned} I(b) = \sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Big(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\times\\ \times\,\eta_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Big(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Big)\,\eta_{00}\,d^3x. \end{aligned}\tag{6.14}$$

Similarly, integrating by parts iteratively the second integral (6.13), we get

$$\begin{gathered} I(g)=\sum_{r=1}^{s}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\times\\ \times\,h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{00}\,d^3x. \end{gathered}\tag{6.15}$$

The quantities η_{ij}, h_{ij}, η_i, h_i, η_{00}, h_{00} in the formulas (6.7), (6.9), (6.11), (6.12), (6.14), and (6.15) should be replaced with the following ones:

$$\eta_{ij}=\frac{\partial b_{ij}}{\partial t},\qquad\qquad \eta_i=\frac{\partial W_i}{\partial t},\tag{6.16}$$

$$h_{ij}=2\,c_{\text{gr}}\,b_{ij},\qquad\qquad h_i=W_i,\tag{6.17}$$

$$h_{00}=c_{\text{gr}}\,b_{00},\qquad\qquad \eta_{00}=\frac{\partial b_{00}}{\partial t}.\tag{6.18}$$

The formulas (6.16), (6.17), and (6.18) are derived by comparing (5.14) with the formulas (5.7), (5.8), and (5.10).

The last step in calculating the components of $\mathbf{J}$ consists in collecting boundary terms from (6.7), (6.9), (6.11), (6.12), (6.14), (6.15) into one formula. Assume N to be the maximal order of the partial derivatives of the form (6.1), (6.2), (6.3) in $\mathcal{L}$. Then from (6.7), (6.9), (6.11), (6.12), (6.14), (6.15), and (5.17) we derive

$$\begin{gathered} \bigl(\mathcal{J}^1\,dx^2\wedge dx^3+\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\sqrt{\det g}\,=\\ =\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\ldots i_s]}\times\\ \times\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\times\\ \times\sqrt{\det g}\,\Bigr)\,h_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\sum_{i=1}^{n}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\times\\ \times\sqrt{\det g}\,\Bigr)\,\eta_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\sum_{i=1}^{n}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\times\\ \times\sqrt{\det g}\,\Bigr)\,h_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+ \end{gathered}\tag{6.19}$$

$$
\begin{aligned}
&+\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\times\\
&\times\sqrt{\det g}\,\biggr)\,\eta_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\
&+\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial g_{00}[i_1\ldots i_s]}\,\times\\
&\times\sqrt{\det g}\,\biggr)\,h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3).
\end{aligned}
\tag{6.20}
$$

Note that the formulas (6.16), (6.17), and (6.18) apply to the formula (6.19) continued in (6.20) as well as to the previous formulas (6.4), (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), and (6.15). Note also that the partial differentiation operators of the form

$$
\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}
$$

are omitted in those terms of (6.19) and (6.20) where $r=1$. The same is true for all previous formulas where these operators are used.

7. Energy of the gravitational field and its density.

Note that the Lagrangian of the theory and its density in (2.12) is subdivided into two parts corresponding to the gravitational field and to matter. Therefore, applying (2.12) to (4.1), (4.2), and (4.3), we derive

$$
E(\Omega)=E_{\text{gr}}(\Omega)+E_{\text{mat}}(\Omega).\qquad\qquad \mathcal{H}=\mathcal{H}_{\text{gr}}+\mathcal{H}_{\text{mat}}.
\tag{7.1}
$$

Due to (2.3) and (2.4) we can calculate $\mathcal{H}_{\text{gr}}$ explicitly. The Lagrangian density (2.3) does not depend on Q_i and W_i. Therefore

$$
\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta W_i}\biggr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}=0.
\tag{7.2}
$$

Similarly, the Lagrangian density (2.3) does not depend on b_{00}. Therefore

$$
\biggl(\frac{\delta\mathcal{L}}{\delta b_{00}}\biggr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}=0.
\tag{7.3}
$$

Applying (7.1), (7.2), and (7.3) to (4.2), we derive

$$
\mathcal{H}_{\text{gr}}=\sum_{i=1}^{3}\sum_{j=1}^{3}\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\biggr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\,b_{ij}-\mathcal{L}_{\text{gr}}.
\tag{7.4}
$$

The variational derivative in (7.4) was already calculated in [15], see the formula (4.3) therein. This derivative is given by the following formula:

$$
\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\biggr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}=\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{8\,\pi\,\gamma}\biggl(b^{ij}-\sum_{k=1}^{3}b^k_k\,g^{ij}\biggr).
\tag{7.5}
$$

Applying (2.3), (2.4) and (7.5) to (7.4), we derive

$$\begin{aligned}\mathcal{H}_{\text{gr}} = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\sqrt{g_{00}}\,\biggl(g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b_q^k\,b_k^q - g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b_k^k\,b_q^q - R\,+\\ +\,2\,\Lambda + g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\,g_{00} - \frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\biggr).\end{aligned}\tag{7.6}$$

This is the formula for the density of the gravitational energy. The amount of the gravitational energy enclosed in a domain Ω is given by the integral similar to (4.3):

$$E_{\text{gr}}(\Omega) = \int\limits_{\Omega} \mathcal{H}_{\text{gr}}\,\sqrt{\det g}\,d^3x.\tag{7.7}$$

The density $\mathcal{H}_{\text{gr}}$ in (7.7) is given by the explicit formula (7.6).

8. Energy flow of the gravitational field and its density.

As we noted above the total Lagrangian of the theory (2.12) is subdivided into two parts responsible for the gravitational field and for matter. Therefore the density vector of the total energy flow $\mathbf{J}$ also subdivides into two parts:

$$\mathbf{J} = \mathbf{J}_{\text{gr}} + \mathbf{J}_{\text{mat}}.\tag{8.1}$$

The first term in the formula (8.1) can be calculated explicitly. For this purpose we apply the formula (5.17) replacing $\mathcal{L}$ by $\mathcal{L}_{\text{gr}}$ in it: We know that $\mathcal{L}_{\text{gr}}$ does not depend on Q_i and W_i. Moreover we know the relationship (7.3). This yields

$$\begin{aligned}\hat L_{\text{gr}}(\Omega) = L_{\text{gr}}(\Omega) + \varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\frac{\partial b_{ij}}{\partial t}\,\sqrt{\det g}\,d^3x\,+\\ +\,\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\,2\,c_{\text{gr}}\,b_{ij}\,\sqrt{\det g}\,d^3x\,+\\ +\,\varepsilon\int\limits_{\Omega}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\Bigr)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\,c_{\text{gr}}\,b_{00}\,\sqrt{\det g}\,d^3x + \varepsilon\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}_{\text{gr}}^1\,dx^2\wedge dx^3\,+\\ +\,\mathcal{J}_{\text{gr}}^2\,dx^3\wedge dx^1 + \mathcal{J}_{\text{gr}}^3\,dx^1\wedge dx^2\bigr)\,\sqrt{\det g} + \ldots\,.\end{aligned}\tag{8.2}$$

We know that only the terms with partial derivatives with respect to spacial coordinates x^1, x^2, x^3 produce boundary terms in (8.2). The term with $2\,\Lambda$ in (2.3) does not have spacial derivatives. The terms with $b_q^k\,b_k^q$ and $b_k^k\,b_q^q$. also do not have spacial derivatives. The rest are three terms in (2.4):

1) the term with $\nabla_{kq}\,g_{00}$;
2) the term with $\nabla_k\,g_{00}\,\nabla_q\,g_{00}$;
3) the term with R.

Let's begin with the term containing $\nabla_{kq}\, g_{00}$. This double covariant derivative itself is written using connection components Γ^s_{kq} of the metric (1.3):

$$\nabla_{kq}\, g_{00} = \frac{\partial^2 g_{00}}{\partial x^k\, \partial x^q} - \sum_{s=1}^{3} \Gamma^s_{kq}\, \frac{\partial g_{00}}{\partial x^s}. \tag{8.3}$$

Applying the variation of the metric g_{ij} from (5.14) to Γ^i_{kq} in (8.3), we get

$$\hat{\Gamma}^s_{kq} = \Gamma^s_{kq} + \frac{\varepsilon}{2} \sum_{r=1}^{3} g^{sr}\, \left(\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\right) + \ldots\,. \tag{8.4}$$

Apart from the variation of the metric, we should take into account the variation of the scalar function g_{00} itself in (5.14). As a result we get

$$\begin{gathered}\hat{\nabla}_{kq}\, \hat{g}_{00} = \nabla_{kq}\, g_{00} + \nabla_{kq}\, h_{00} - \\ - \frac{\varepsilon}{2} \sum_{r=1}^{3} \sum_{s=1}^{3} g^{sr} \bigl(\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\bigr)\, \nabla_s\, g_{00} + \ldots\,.\end{gathered} \tag{8.5}$$

The formula (8.5) is derived using (8.4). Due to (8.5) the term (8.3) contributes to the left hand side of (8.2) through the following two integrals:

$$\begin{gathered}L_1 = \frac{c^4_{\text{gr}}\, \varepsilon}{16\, \pi\, \gamma} \int\limits_{\Omega} \frac{g_{00}^{-1/2}}{2} \sum_{k=1}^{3} \sum_{q=1}^{3} \sum_{r=1}^{3} \sum_{s=1}^{3} g^{sr}\, g^{kq}\, \bigl(\nabla_k\, h_{rq} + \\ + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\bigr)\, \nabla_s\, g_{00}\, \sqrt{\det g}\; d^3x,\end{gathered} \tag{8.6}$$

$$L_2 = -\frac{c^4_{\text{gr}}\, \varepsilon}{16\, \pi\, \gamma} \int\limits_{\Omega} g_{00}^{-1/2} \sum_{k=1}^{3} \sum_{q=1}^{3} g^{kq}\, \nabla_{kq}\, h_{00}\, \sqrt{\det g}\; d^3x. \tag{8.7}$$

Now let's proceed to the term with $\nabla_k\, g_{00}\, \nabla_q\, g_{00}$ in (2.4). The covariant derivatives in this term do not use the connection components Γ^s_{kq}. Therefore this term contributes to the left hand side of the formula (8.2) through the following integral:

$$L_3 = \frac{c^4_{\text{gr}}\, \varepsilon}{16\, \pi\, \gamma} \int\limits_{\Omega} g_{00}^{-3/2} \sum_{k=1}^{3} \sum_{q=1}^{3} g^{kq}\, \nabla_k\, g_{00}\, \nabla_q\, h_{00}\, \sqrt{\det g}\; d^3x. \tag{8.8}$$

The term with the scalar curvature R in (2.4) is the most complicated to handle. Applying the variation of the metric g_{ig} from (5.14) to R, we get the formula

$$\hat{R} = R - \varepsilon \sum_{i=1}^{3} \sum_{j=1}^{3} R^{ij}\, h_{ij} + \varepsilon \sum_{k=1}^{3} \nabla_k Z^k + \ldots\,, \tag{8.9}$$

where the following notations are introduced:

$$Z^k = \sum_{q=1}^{3} \sum_{j=1}^{3} \bigl(g^{jq}\, Y^k_{jq} - g^{kq}\, Y^j_{jq}\bigr), \tag{8.10}$$

$$Y^s_{kq} = \frac{1}{2} \sum_{r=1}^{3} g^{sr}\, \left(\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\right). \tag{8.11}$$

The formulas (8.4), (8.9), (8.10), and (8.11) are derived in [15]. Due to (8.9) the term with R contributes to the left hand side of (8.2) through the integral

$$L_4=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega} g_{00}^{1/2}\sum_{k=1}^{3}\nabla_k Z^k\,\sqrt{\det g}\;d^3x. \tag{8.12}$$

Now we return to the integral (8.6). This integral can be simplified and written as a sum of two integrals. Here are these two integrals:

$$\begin{aligned}L_1=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}2\,\nabla_k\,h^{sk}\,\nabla_s\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x\,-\\-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}\sum_{r=1}^{3}g^{sr}\,\nabla_r\,h^k_k\,\nabla_s\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{8.13}$$

Applying the integration by parts to the integrals (8.13), we get

$$\begin{aligned}L_1=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}2\,h^{sk}\,\nabla_s\bigl(g_{00}^{1/2}\bigr)\,n_k\,dS\,-\\-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}2\,h^{sk}\,\nabla_{sk}\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x\,-\\-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}\sum_{r=1}^{3}g^{sr}\;h^k_k\,\nabla_s\bigl(g_{00}^{1/2}\bigr)\,n_r\,dS\,+\\+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}\sum_{r=1}^{3}g^{sr}\,h^k_k\,\nabla_{sr}\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{8.14}$$

Here in (8.14) through dS we denote the infinitesimal area element on the boundary $\partial\,\Omega$ of the domain Ω, while n_k and n_r are the covariant components of the unit normal vector $\mathbf{n}$ on the boundary $\partial\,\Omega$ directed toward the outside of the domain Ω.

Proceeding to the integral (8.7), we integrate it by parts twice. This yields

$$\begin{aligned}L_2=-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}g_{00}^{-1/2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_q\,h_{00}\,n_k\,dS\,+\\+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,n_q\,dS\,-\\-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_{kq}\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{8.15}$$

In (8.15), like in (8.14), dS is the infinitesimal area element of $\partial\,\Omega$, while n_k and n_q are the covariant components of the unit normal vector $\mathbf{n}$ on $\partial\,\Omega$.

Now we proceed to the integral (8.8). This integral can be written as

$$L_3=-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\bigl(g_{00}^{-1/2}\bigr)\,\nabla_q\,h_{00}\,\sqrt{\det g}\;d^3x. \tag{8.16}$$

Integrating (8.16) by parts, we derive the following formula:

$$\begin{aligned}L_3=&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,n_q\,dS\,+\\&+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_{kq}\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{8.17}$$

The next step is to handle the integral (8.12). Integrating by parts in it, we get

$$\begin{aligned}L_4=&\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}g_{00}^{1/2}\sum_{k=1}^{3}Z^k\,n_k\,dS\,-\\&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\nabla_k\bigl(g_{00}^{1/2}\bigr)\,Z^k\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{8.18}$$

Applying (8.11) to (8.10), the following formula is derived:

$$Z^k=\sum_{q=1}^{3}\nabla_q\,h^{kq}-\sum_{q=1}^{3}\sum_{r=1}^{3}g^{kq}\,\nabla_q\,h^r_r. \tag{8.19}$$

The formula (8.19) can be found in [15]. We apply this formula (8.19) to the second integral in (8.18). As a result we obtain the formula

$$\begin{aligned}L_4=&\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}g_{00}^{1/2}\sum_{k=1}^{3}Z^k\,n_k\,dS\,-\\&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\nabla_k\bigl(g_{00}^{1/2}\bigr)\,\nabla_q\,h^{kq}\sqrt{\det g}\;d^3x\,+\\&+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_k\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,\nabla_q\,h^r_r\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{8.20}$$

Then we apply integration by parts to the second and third integrals in (8.20):

$$\begin{aligned}L_4=&\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}g_{00}^{1/2}\sum_{k=1}^{3}Z^k\,n_k\,dS\,-\\&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\nabla_k\bigl(g_{00}^{1/2}\bigr)\,h^{kq}\,n_q\,dS\,+\\&+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\nabla_{kq}\bigl(g_{00}^{1/2}\bigr)\,h^{kq}\sqrt{\det g}\;d^3x\,+\end{aligned} \tag{8.21}$$

$$
\begin{aligned}
&+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_k\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,h^r_r\,n_q\,dS\,-\\
&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_{kq}\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,h^r_r\,\sqrt{\det g}\,d^3x.
\end{aligned}
\tag{8.22}
$$

The formula (8.21) is continued in (8.22). Like in the previous formulas, in (8.21) and (8.22) through dS we denote the infinitesimal area element on the boundary $\partial\,\Omega$ of the domain Ω, while n_k and n_q are the covariant components of the unit normal vector $\mathbf{n}$ on the boundary $\partial\,\Omega$ directed toward the outside of the domain Ω.

Now let's recall that the surface integral of the second kind in (8.2) can be transformed to a surface integral of the first kind:

$$
\begin{aligned}
&\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}_{\text{gr}}^1\,dx^2\wedge dx^3+\mathcal{J}_{\text{gr}}^2\,dx^3\wedge dx^1\,+\\
&+\mathcal{J}_{\text{gr}}^3\,dx^1\wedge dx^2\bigr)\,\sqrt{\det g}\,=\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\mathcal{J}^k\,n_k\,dS.
\end{aligned}
\tag{8.23}
$$

Due to (8.23) we can put together the boundary integrals from (8.14), (8.15), (8.17), (8.21), and (8.22) and compare their sum with the right hand side of (8.23):

$$
\begin{aligned}
&\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\biggl(\sum_{s=1}^{3}2\,h^{sk}\,\nabla_s\bigl(g_{00}^{1/2}\bigr)-\sum_{q=1}^{3}\sum_{s=1}^{3}g^{sk}\ h^q_q\,\nabla_s\bigl(g_{00}^{1/2}\bigr)\,-\\
&-\sum_{q=1}^{3}g_{00}^{-1/2}\,g^{kq}\,\nabla_q\,h_{00}+\sum_{q=1}^{3}g^{kq}\,\nabla_q\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,-\\
&-\sum_{q=1}^{3}g^{kq}\,\nabla_q\bigl(\,g_{00}^{-1/2}\bigr)\,h_{00}+g_{00}^{1/2}\,Z^k-\sum_{q=1}^{3}\nabla_q\bigl(g_{00}^{1/2}\bigr)\,h^{kq}\,+\\
&+\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_q\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,h^r_r=\varepsilon\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\mathcal{J}_{\text{gr}}^k\,n_k\,dS.
\end{aligned}
\tag{8.24}
$$

Applying (8.19), from (8.24) we determine the components of the vector $\mathbf{J}_{\text{gr}}$:

$$
\begin{aligned}
\mathcal{J}_{\text{gr}}^k=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\biggl(&\sum_{i=1}^{3}g_{00}^{1/2}\,\nabla_i\,h^{ik}-\sum_{i=1}^{3}\sum_{q=1}^{3}g_{00}^{1/2}\,g^{ik}\,\nabla_i\,h^q_q\,+\\
&+\sum_{i=1}^{3}h^{ik}\,\nabla_i\bigl(g_{00}^{1/2}\bigr)-\sum_{i=1}^{3}g_{00}^{-1/2}\,g^{ik}\,\nabla_i\,h_{00}\biggr).
\end{aligned}
\tag{8.25}
$$

The ultimate formula for $\mathcal{J}^k$ is produced from the formula (8.25) by applying the formulas (6.17) and (6.18) to it. This yields

$$
\begin{aligned}
\mathcal{J}_{\text{gr}}^k=\frac{c_{\text{gr}}^5}{16\,\pi\,\gamma}\biggl(&\sum_{i=1}^{3}2\,g_{00}^{1/2}\,\nabla_i\,b^{ik}-\sum_{i=1}^{3}\sum_{q=1}^{3}2\,g_{00}^{1/2}\,g^{ik}\,\nabla_i\,b^q_q\,+\\
&+\sum_{i=1}^{3}2\,b^{ik}\,\nabla_i\bigl(g_{00}^{1/2}\bigr)-\sum_{i=1}^{3}g_{00}^{-1/2}\,g^{ik}\,\nabla_i\,b_{00}\biggr).
\end{aligned}
\tag{8.26}
$$

The flow of the gravitational energy through a surface S is given by the formula

$$E(S)=\int\limits_S\sum_{k=1}^{3}\mathcal{J}_{\text{gr}}^{\,k}\,n_k\,dS. \tag{8.27}$$

The components $\mathcal{J}^k$ of the vector $\mathbf{J}_{\text{gr}}$ in (8.27) are given by the formula (8.26).

9. Concluding remarks.

Theorem 5.1 states the total energy conservation law within the second version of the **3D-brane universe model** that does not use the equidistance postulate 1.1. The total energy is the sum of the gravitational energy and the energy of matter. Theorem 5.1 is complemented by the formulas for the density of the total energy (see (4.2)) and for the density of its flow (see (6.19) and (6.20)). The formulas for the density of the gravitational energy and for the density of its flow are explicit (see (7.6) and (8.26)). The formulas (4.2), (6.19), (6.20), (7.6), and (8.26) along with Theorem 5.1 constitute the main result of the present paper. For $g_{00}=1$ the formula (7.6) reduces to the formula (4.58) in [5], while the formula (8.26) reduces to the formula (4.48) in [5].

10. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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