RELATIVISTIC ELASTICITY IN THE 3D-BRANE UNIVERSE MODEL. PART 1.

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Abstract. 3D-brane universe model is the name of a new non-Einsteinian theory of gravity. Within this theory general relativistic nonlinear elastic media are considered. The effect of the relativistic contraction of solid bodies along the direction of their motion known from Einstein's theory is reproduced within the new theory.

1. Introduction.

Einstein's special relativity and his general relativity both are based on the concept of a four-dimensional spacetime which is also known as the block-universe, see [1]. Unlike this, in the 3D-brane universe model the universe is an evolving three-dimensional continuum. In order to maintain continuity with Einstein's theory, the evolution of this three-dimensional continuum is described as the motion of a 3D-brane in a four-dimensional spacetime. However, unlike Einstein's theory, in the new theory the four-dimensional spacetime is not a physical continuum. It is just a mathematical abstraction that has no physical existence.

The new theory of gravity that we follow in this paper has two versions. The first version is developed using the so-called equidistance postulate (see e-prints [2–7] and conference abstracts [8–12]). In the second version of the theory the equidistance postulate is omitted (see e-prints [13–18] and conference abstracts [19–26]). Therefore the second version is more general and we refer the reader to it rather than to the first version. We also refer the reader to the book [27] where the results of e-prints [13–16] are summarized.

There are two phenomena in Einstein's theory of relativity. The first one is the time dilation for moving observers (see [28]) and the second one is the contraction of solid bodies along the direction of their motion (see [29]). The first phenomenon was reproduced within the new theory in [18] using ticks of spring-mass systems as timestamps. The goal of the present paper is to reproduce the second phenomenon within this new theory whose name is 3D-brane universe model.

2. Some preliminaries.

The spacetime of the standard Einstein's general relativity is equipped with three geometric structures: 1) the four-dimensional metric of the signature $(+,-,-,-)$, 2) the orientation that distinguishes the right oriented coordinate systems from the

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left oriented ones, 3) the polarization that distinguished the future light cone from the past light cone at each point (see § 3 of Chapter III in [30]). In the 3D-brane universe model the spacetime is equipped with the auxiliary geometric structure 4) the foliation of 3D-branes. It subdivides the whole spacetime into the union of non-intersecting spacelike 3D-branes except for possibly one point that corresponds to the Big Bang (see [13]). Each 3D-brane of this foliation represents some instantaneous state of the real three-dimensional universe in its evolution pathway. If we assign some specific numeric value t to each 3D-brane in the spacetime that grows from the past to the future according to the polarization, we get a dedicated time variable t in the spacetime which is called *a brane time*.

Dedicated spacial coordinates x^1, x^2, x^3 are also defined with the use of the the foliation of 3D-branes in the spacetime. They are first chosen as local coordinates in some particular 3D-brane. Then they are extended to other 3D-branes along integral curves of the vector field $\mathbf{n}$ of unit vectors normal to the branes. Coordinates x^1, x^2, x^3 introduced in this way are called *spacial comoving coordinates*. They are usually complemented with the time coordinate

$$x^0 = c_{\text{gr}}\, t, \tag{2.1}$$

where t is some brane time variable and c_{gr} is a speed constant analogous to the speed of light. There are several speed constants in the 3D-brane universe model:

$$c_{\text{el}}, \qquad c_{\text{gr}}, \qquad c_{\text{br}}, \qquad c_{\text{nb}}. \tag{2.2}$$

The first constant (2.2) is the regular speed of light (see [31]):

$$c_{\text{el}} = 299792458 \text{ m/s}. \tag{2.3}$$

Generally speaking, the speed of light is an experimentally measured constant that should have an approximate value. However, in 1983 the 17th meeting of the General Conference on Weights and Measures has redefined the value of one meter in such a way that the speed of light has got its exact value (2.3). Since 1967 the time unit of one second (see [32]) in (2.3) is defined to be exactly 9192631770 cycles of the hyperfine structure transition frequency of caesium-133 atoms.

The second speed constant in (2.2) is the speed of gravity, the third constant is the critical speed of baryonic matter, and the fourth constant is the critical speed of non-baryonic dark matter. Actually dark matter can be subdivided into several sorts each with its own critical speed.

In the standard Einstein's relativity all of the speed constants (2.2) are equal to each other. In the 3D-brane universe model they are potentially different. The only equality established thus far is the following one:

$$c_{\text{gr}} = c_{\text{br}}. \tag{2.4}$$

For the proof of the equality (2.4) see § 4 of Chapter IV in the book [27]).

Spacial comoving coordinates x^1, x^2, x^3 complemented with the brane time coordinate (2.1) form a complete set of coordinates in the four-dimensional spacetime

$$x^0 = c_{\text{gr}}\, t, \qquad x^1, \qquad x^2, \qquad x^3. \tag{2.5}$$

In such coordinates (2.5) the four-dimensional metric of the four-dimensional space-time takes the following block-diagonal form:

$$\begin{Vmatrix} g_{00} & 0 & 0 & 0 \\ 0 & -g_{11} & -g_{12} & -g_{13} \\ 0 & -g_{21} & -g_{22} & -g_{23} \\ 0 & -g_{31} & -g_{32} & -g_{33} \end{Vmatrix}. \tag{2.6}$$

The metric (2.6) comprises the scalar function

$$g_{00}(t, x^1, x^2, x^3), \tag{2.7}$$

and the time-dependent three-dimensional metric with the components

$$g_{ij} = g_{ij}(t, x^1, x^2, x^3), \quad 1 \leqslant i, j \leqslant 3. \tag{2.8}$$

The positive scalar function (2.7) and the Riemannian metric (2.8) both describe the gravitational field in the real three-dimensional universe. Here are the differential equations for the scalar function (2.7) and for the metric (2.8):

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k + \frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q + \frac{R}{2}\, g_{00} - \Lambda\, g_{00} = \frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\, g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}, \tag{2.9}$$

$$\begin{gathered} \frac{g_{00}^{-2}}{2\,c_{\text{gr}}}\left(\sum_{k=1}^{3} b^k_k\, g_{ij} - b_{ij}\right)\dot g_{00} + \frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(g^{kq}\, g_{ij} - \delta^k_i\,\delta^q_j\bigr)\,\nabla_{kq}\, g_{00} - \\ -\frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(g^{kq}\, g_{ij} - \delta^k_i\,\delta^q_j\bigr)\,\nabla_k\, g_{00}\,\nabla_q\, g_{00} + g_{00}^{-1}\left(\frac{1}{c_{\text{gr}}}\,\dot b_{ij} - \right. \\ -\sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot b^k_k\, g_{ij} - \sum_{k=1}^{3}(b_{ki}\, b^k_j + b_{kj}\, b^k_i) - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k\,\frac{g_{ij}}{2} - \\ \left. -\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q\,\frac{g_{ij}}{2} + \sum_{k=1}^{3} b^k_k\, b_{ij}\right) + R_{ij} - \frac{R}{2}\, g_{ij} + \Lambda\, g_{ij} = -\frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\, g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}. \end{gathered} \tag{2.10}$$

In (2.9) and (2.10) the following notations are used:

$$\dot g_{00} = \frac{\partial g_{00}}{\partial t}, \qquad\qquad b_{ij} = \frac{\dot g_{ij}}{2\,c_{\text{gr}}} = \frac{1}{2\,c_{\text{gr}}}\,\frac{\partial g_{ij}}{\partial t} = \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}. \tag{2.11}$$

Through ∇ in (2.10) we denote covariant derivatives with respect to the metric connection associated with the three-dimensional metric (2.8), R_{ij} in (2.10) are the components of the Ricci tensor associated with this metric and R in (2.9) and (2.10) is the scalar curvature of this metric. Apart from (2.11), we see two constants Λ and γ, where γ is Newton's gravitational constant (see [33]):

$$\gamma \approx\ 6.674\cdot 10^{-8}\ cm^3\ \cdot\ g^{-1}\ \cdot\ s^{-2}. \tag{2.12}$$

Through Λ in (2.9) and (2.10) we denote the cosmological constant

$$\Lambda \approx -10^{-56}\ cm^{-2}. \tag{2.13}$$

Unlike that of [34], we use the cosmological constant with negative sign. This is done in order to fit notations used in the book [30]. Apart from (2.11), (2.12), and (2.13), in (2.9) and (2.10) we see the function $\mathcal{L}_{\text{mat}}$ and its variational derivatives

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}, \qquad\qquad \frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}. \tag{2.14}$$

The function $\mathcal{L}_{\text{mat}}$ in (2.14) is the Lagrangian density of matter. As matter in this paper we choose a moving elastic solid medium.

3. Relativistic deformation tensor.

Assume that some solid medium was melt and frozen again in a three-dimensional space with the coordinates y^1, y^2, y^3 and with the three-dimensional metric

$$\eta_{ij} = \eta_{ij}(y^1, y^2, y^3), \quad 1 \leqslant i, j \leqslant 3. \tag{3.1}$$

Then assume that this medium is immersed into the real universe which is described by some comoving coordinates x^1, x^2, x^3, some brane time t, the scalar function (2.7), and the metric (2.8). The immersion is described by the functions

$$\begin{cases} x^1 = x^1(t, y^1, y^2, y^3),\\ x^2 = x^2(t, y^1, y^2, y^3),\\ x^3 = x^3(t, y^1, y^2, y^3). \end{cases} \tag{3.2}$$

The functions (3.2) are assumed to produce an invertible mapping whose inverse mapping is given by similar three functions

$$\begin{cases} y^1 = y^1(t, x^1, x^2, x^3),\\ y^2 = y^2(t, x^1, x^2, x^3),\\ y^3 = y^3(t, x^1, x^2, x^3). \end{cases} \tag{3.3}$$

Differentiating the functions (3.2) with respect to the time variable t, we get the components of the velocity vector $\mathbf{v}$ of our medium in the real universe:

$$v^i = \dot{x}^i = \frac{\partial x^i(t, y^1, y^2, y^3)}{\partial t}, \quad 1 \leqslant i \leqslant 3. \tag{3.4}$$

In order to express the components of the velocity vector $\mathbf{v}$ through the coordinates x^1, x^2, x^3 in the real universe we need to substitute (3.3) into the arguments of the functions (3.4). As a result we get three functions

$$v^i = v^i(t, x^1, x^2, x^3), \quad 1 \leqslant i \leqslant 3. \tag{3.5}$$

The quantities (3.5) are interpreted as the components of the flow velocity vector of our medium flow in the real universe (see [35]).

Using (3.3), we can transfer the metric (3.1) to the real universe

$$G_{ij}=\sum_{r=1}^{3}\sum_{s=1}^{3}\eta_{rs}\,\frac{\partial y^r}{\partial x^i}\,\frac{\partial y^s}{\partial x^j},\quad 1\leqslant i,j\leqslant 3. \tag{3.6}$$

In classical physics the metric (3.6) is the most comfortable metric for the medium. This means that if $g_{ij}=G_{ij}$, then the immersion (3.2) does not produce any stress in the medium. Therefore in classical physics the nonlinear deformation tensor is defined by means of the following formula:

$$u_{ij}=\frac{g_{ij}-G_{ij}}{2}. \tag{3.7}$$

The formula (3.7) coincides with the formula (4.11) in [36].

Relativistic physics is different. Einstein's theory of relativity predicts relativistic contraction of moving rods along the direction of their motion. In order to maintain continuity with Einstein's theory we include this phenomenon into the 3D-brane universe model. For this purpose we define the contraction operator with the matrix

$$C^i_j=\delta^i_j-\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(1-\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c^2_{\text{br}}}}\,\right). \tag{3.8}$$

Through δ^i_j here we denote the Kronecker delta (see [37]). The inverse operator to (3.8) is the relativistic extension operator. Its matrix is given by the formula

$$E^i_j=\delta^i_j-\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(1-\frac{1}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c^2_{\text{br}}}}}\right). \tag{3.9}$$

Now we apply the operator (3.9) to the metric (2.8) for to produce a new metric:

$$\check{g}_{ij}=\sum_{r=1}^{3}\sum_{s=1}^{3}E^r_i\,g_{rs}\,E^s_j,\quad 1\leqslant i,j\leqslant 3. \tag{3.10}$$

The metric (3.10) is used in the following formula:

$$u_{ij}=\frac{\check{g}_{ij}-G_{ij}}{2}. \tag{3.11}$$

The formula (3.11) is analogous to the classical formula (3.7). It defines *the relativistic deformation tensor.*

4. Deformation-free steady flow.

The term *steady flow* means a time independent non-changing flow. Here we consider a flow with constant velocity $\mathbf{v}$ in (3.4):

$$v^i=\text{const},\quad 1\leqslant i\leqslant 3. \tag{4.1}$$

Then we choose flat spaces with the following data in (2.7), (2.8), and (3.1):

$$g_{00}=1, \qquad\qquad g_{ij}=\eta_{ij}=\delta_{ij}. \tag{4.2}$$

Through δ_{ij} in (4.2) we again denote the Kronecker delta (see [37]). Applying (4.1) to (3.4), we derive the following formula for the immersion functions (3.2):

$$x^i=v^i\,t+x^i(y^1,y^2,y^3). \tag{4.3}$$

A deformation-free flow means that $u_{ij}=0$ in (3.11). From $u_{ij}=0$ we derive

$$\check{g}_{ij}=G_{ij} \tag{4.4}$$

Applying (4.2), (3.6), and (3.10) to (4.4), we get the equalities

$$\sum^3_{r=1}\sum^3_{s=1}\frac{\partial y^r}{\partial x^i}\,\delta_{rs}\,\frac{\partial y^s}{\partial x^j}=\sum^3_{r=1}\sum^3_{s=1}E^r_i\,\delta_{rs}\,E^s_j. \tag{4.5}$$

The equalities (4.5) can be satisfied identically if we set

$$\frac{\partial y^r}{\partial x^i}=E^r_i,\quad 1\leqslant i,r\leqslant 3. \tag{4.6}$$

The equalities (4.6) constitute a system of partial differential equations. These differential equations are consistent since

$$\frac{\partial^2 y^r}{\partial x^i\partial x^j}=\frac{\partial E^r_i}{\partial x^j}=0=\frac{\partial E^r_j}{\partial x^i}=\frac{\partial^2 y^r}{\partial x^j\partial x^i}.$$

Moreover, the equations (4.6) are easily solvable:

$$y^r=\sum^3_{i=1}E^r_i\,x^i+E^r(t,\mathbf{v}). \tag{4.7}$$

Applying the operator (3.8) to both sides of (4.7), we get

$$x^i=\sum^3_{r=1}C^i_r\,y^r-\sum^3_{r=1}C^i_r\,E^r(t,\mathbf{v}). \tag{4.8}$$

Comparing (4.8) with (4.3), we can write

$$x^i=v^i\,t+\sum^3_{r=1}C^i_r\,y^r+x^i_0, \tag{4.9}$$

where $x^i_0=\text{const}$. The relationships (4.9) are explicit forms of the immersion formulas (3.2) for a deformation-free steady flow. The term with the operator (3.8) in the right hand side of (4.9) means that the immersion (4.9) performs the relativistic contraction of a medium along the direction of the velocity vector $\mathbf{v}$ with the constant components (4.1).

5. Differential equation for the rest mass density.

Let's return to the solid medium that was melt and then frozen again in a three-dimensional space with the coordinates y^1, y^2, y^3 and with the three-dimensional metric (3.1). This medium is at rest in that space. Let

$$\rho_0(y^1, y^2, y^3) \tag{5.1}$$

be its rest mass density. Upon being immersed into the real three-dimensional universe with the coordinates x^1, x^2, x^3 and the three-dimensional metric (2.8) our medium gets a new rest mass density

$$\rho(t, x^1, x^2, x^3). \tag{5.2}$$

The arguments of the functions (5.1) and (5.2) are related to each other through the formulas (3.2) and (3.3). Let $\Omega(y)$ be some domain in the space with the coordinates y^1, y^2, y^3 and let $\Omega(x)$ be its image under the mapping (3.2) for some fixed value of t. Then by definition we have the following integral relationship:

$$\int\limits_{\Omega(y)} \rho_0\,\sqrt{\det\eta}\,d^3y = \int\limits_{\Omega(x)} \rho\,\sqrt{\det g}\,d^3x. \tag{5.3}$$

Taking into account (3.6), we can transform the left integral (5.3) as follows:

$$\int\limits_{\Omega(y)} \rho_0\,\sqrt{\det\eta}\,d^3y = \int\limits_{\Omega(x)} \rho_0\,\sqrt{\det\eta}\,|\det J|\,d^3x = \int\limits_{\Omega(x)} \rho_0\,\sqrt{\det G}\,d^3x. \tag{5.4}$$

Here J is the Jacobian matrix (see [38]) with the components

$$J^i_j = \frac{\partial y^i}{\partial x^j}. \tag{5.5}$$

Comparing (5.3) and (5.4) we derive the formula

$$\rho = \rho(t, x^1, x^2, x^3) = \rho_0\,\frac{\sqrt{\det G}}{\sqrt{\det g}}. \tag{5.6}$$

The time variable t is complementary to the spacial variables y^1, y^2, y^3 in (3.2) and the same time variable t is complementary to the spacial variables x^1, x^2, x^3 in (3.3). For the sake of convenience in further calculations we introduce the second time variable $\tau = t$ and write (3.2) and (3.3) as follows:

$$\begin{cases} t = \tau,\\ x^1 = x^1(\tau, y^1, y^2, y^3),\\ x^2 = x^2(\tau, y^1, y^2, y^3),\\ x^3 = x^3(\tau, y^1, y^2, y^3), \end{cases} \qquad \begin{cases} \tau = t,\\ y^1 = y^1(t, x^1, x^2, x^3),\\ y^2 = y^2(t, x^1, x^2, x^3),\\ y^3 = y^3(t, x^1, x^2, x^3). \end{cases} \tag{5.7}$$

From now on we assume τ to be complementary to the variables y^1, y^2, y^3 and t to be complementary to the variables x^1, x^2, x^3. The formula (3.4) for the components of the velocity vector $\mathbf{v}$ then is rewritten as

$$v^i=\dot{x}^i(\tau,y^1,y^2,y^3)=\frac{\partial x^i(\tau,y^1,y^2,y^3)}{\partial\tau},\quad 1\leqslant i\leqslant 3,\tag{5.8}$$

while the formula (3.5) remains unchanged.

The density ρ in (5.2) and (5.6) is a function in the real universe. Its arguments are t, x^1, x^2, x^3. We shall use (5.6) in order to calculate the time derivative of the density (5.2). However, before doing it we calculate the following divergency:

$$\operatorname{div}(\rho\,\mathbf{v})=\sum_{r=1}^{3}\nabla_r(\rho\,v^r)=\frac{1}{\sqrt{\det g}}\sum_{r=1}^{3}\frac{\partial(\rho\,v^r\sqrt{\det g})}{\partial x^r}.\tag{5.9}$$

In deriving (5.9) we used the formula (4.7) from Chapter IV in [30]. Applying the formula (5.6) to the formula (5.9), we derive

$$\operatorname{div}(\rho\,\mathbf{v})=\frac{1}{\sqrt{\det g}}\sum_{r=1}^{3}\frac{\partial(\rho_0\,v^r\sqrt{\det G})}{\partial x^r}.\tag{5.10}$$

The formula (5.10) can be rewritten as follows:

$$\operatorname{div}(\rho\,\mathbf{v})=\frac{1}{\sqrt{\det g}}\sum_{r=1}^{3}v^r\,\frac{\partial(\rho_0\sqrt{\det G})}{\partial x^r}+\rho\sum_{r=1}^{3}\frac{\partial v^r}{\partial x^r}.\tag{5.11}$$

Applying the formula (5.8) to the last term in (5.11), we derive

$$\begin{aligned}\sum_{r=1}^{3}\frac{\partial v^r}{\partial x^r}&=\sum_{r=1}^{3}\sum_{m=1}^{3}\frac{\partial v^r}{\partial y^m}\,\frac{\partial y^m}{\partial x^r}=\sum_{r=1}^{3}\sum_{m=1}^{3}\frac{\partial^2 x^r}{\partial y^m\,\partial\tau}\,\frac{\partial y^m}{\partial x^r}=\\&=\sum_{r=1}^{3}\sum_{m=1}^{3}\frac{\partial^2 x^r}{\partial\tau\,\partial y^m}\,\frac{\partial y^m}{\partial x^r}=\sum_{r=1}^{3}\sum_{m=1}^{3}\frac{\partial}{\partial\tau}\Bigl(\frac{\partial x^r}{\partial y^m}\Bigr)\,\frac{\partial y^m}{\partial x^r}.\end{aligned}\tag{5.12}$$

Let's denote through M the Jacobian matrix with the components

$$M^r_m=\frac{\partial x^r}{\partial y^m}.\tag{5.13}$$

The matrix M with the components (5.13) is inverse to the matrix J with the components (5.5). Therefore (5.12) is rewritten as

$$\sum_{r=1}^{3}\frac{\partial v^r}{\partial x^r}=\sum_{r=1}^{3}\sum_{m=1}^{3}\frac{\partial M^r_m}{\partial\tau}\,J^m_r=\operatorname{tr}\Bigl(M^{-1}\cdot\frac{\partial M}{\partial\tau}\Bigr).\tag{5.14}$$

The next step is to apply Jacobi's formula for differentiating determinants (see [39]) to the right hand side of (5.14). This yields

$$\sum_{r=1}^{3}\frac{\partial v^r}{\partial x^r}=\operatorname{tr}\Bigl(M^{-1}\cdot\frac{\partial M}{\partial\tau}\Bigr)=\frac{1}{\det M}\,\frac{\partial(\det M)}{\partial\tau}.\tag{5.15}$$

Due to (5.15) the formula (5.11) ultimately transforms to

$$\operatorname{div}(\rho\,\mathbf{v})=\frac{1}{\sqrt{\det g}}\sum_{r=1}^{3}v^r\,\frac{\partial(\rho_0\sqrt{\det G})}{\partial x^r}+\frac{\rho}{\det M}\,\frac{\partial(\det M)}{\partial\tau}. \tag{5.16}$$

Now, as we planned above, using (5.6), we calculate the partial derivative

$$\frac{\partial\rho}{\partial t}=\frac{1}{\sqrt{\det g}}\,\frac{\partial(\rho_0\,\sqrt{\det G})}{\partial t}-\frac{\rho}{2\,\det g}\,\frac{\partial(\det g)}{\partial t}. \tag{5.17}$$

The second term in the right hand side of (5.17) is similar to the second term in the right hand side of (5.16). Again, applying Jacobi's formula for differentiating determinants (see [39]) and taking into account (2.11), we derive

$$\frac{\partial\rho}{\partial t}=\frac{1}{\sqrt{\det g}}\,\frac{\partial(\rho_0\,\sqrt{\det G})}{\partial t}-c_{\mathrm{gr}}\,\rho\sum_{k=1}^{3}b^k_k. \tag{5.18}$$

The next step is to add two formulas (5.16) and (5.18). This yields

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\,\mathbf{v})=\frac{1}{\sqrt{\det g}}\left(\frac{\partial}{\partial t}+\sum_{r=1}^{3}v^r\,\frac{\partial}{\partial x^r}\right)(\rho_0\,\sqrt{\det G})\,+\\+\,\frac{\rho}{\det M}\,\frac{\partial(\det M)}{\partial\tau}-c_{\mathrm{gr}}\,\rho\sum_{k=1}^{3}b^k_k.\end{gathered} \tag{5.19}$$

Using (5.7) and (5.8), one can easily derive that

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial t}+\sum_{r=1}^{3}v^r\,\frac{\partial}{\partial x^r}. \tag{5.20}$$

Then, applying (5.20) to (5.19), we get

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\,\mathbf{v})=\frac{1}{\sqrt{\det g}}\,\frac{\partial(\rho_0\,\sqrt{\det G})}{\partial\tau}\,+\\+\frac{\rho}{\det M}\,\frac{\partial(\det M)}{\partial\tau}-c_{\mathrm{gr}}\,\rho\sum_{k=1}^{3}b^k_k.\end{gathered} \tag{5.21}$$

From (3.6) and (5.5) we derive the following relationship for determinants:

$$\det G=\det\eta\,(\det J)^2. \tag{5.22}$$

Actually we have already used the relationship (5.22) in (5.4). Now we apply it once more in order to transform (5.21). It yields

$$\frac{1}{\sqrt{\det g}}\,\frac{\partial(\rho_0\,\sqrt{\det G})}{\partial\tau}=\frac{1}{\sqrt{\det g}}\,\frac{\partial(\rho_0\,\sqrt{\det\eta})}{\partial\tau}\,|\det J|+\frac{\rho}{|\det J|}\,\frac{\partial|\det J|}{\partial\tau}. \tag{5.23}$$

Note that ρ_0 in (5.1) and the components of the matrix η in (3.1) depend on y^1, y^2, y^3, but they do not depend on τ. Therefore (5.23) reduces to

$$\frac{1}{\sqrt{\det g}}\,\frac{\partial(\rho_0\sqrt{\det G})}{\partial\tau}=\frac{\rho}{|\det J|}\,\frac{\partial|\det J|}{\partial\tau}. \tag{5.24}$$

The right hand side of (5.24) and the second term in the right hand side of (5.21) both comprise logarithmic derivatives. Therefore, applying (5.24) to (5.21), we get

$$\frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\,\mathbf{v})=\rho\,\frac{\partial\ln(|\det M|\cdot|\det J|)}{\partial\tau}-c_{\text{gr}}\,\rho\sum_{k=1}^{3}b^k_k. \tag{5.25}$$

The Jacobian matrices J and M with the components (5.5) and (5.13) are inverse to each other. Therefore we have the equality

$$|\det M|\cdot|\det J|=1. \tag{5.26}$$

Applying the equality (5.26) to the equality (5.25), we derive the following differential equation for the rest mass density ρ in (5.2) and (5.6):

$$\frac{\partial\rho}{\partial t}+\sum_{k=1}^{3}c_{\text{gr}}\,\rho\,b^k_k+\sum_{i=1}^{3}\nabla_i(\rho\,v^i)=0. \tag{5.27}$$

The differential equation (5.27) is similar to the differential equation (5.23) in [16] expressing the total energy conservation law (see also § 8 of Chapter III in [27]). Like (5.23) in [16], the equation (5.27) can be transformed to an integral equation:

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\rho\sqrt{\det g}\;d^3x+\int\limits_{\partial\,\Omega}\sum_{i=1}^{3}\rho\,v^i\,n_i\,dS=0. \tag{5.28}$$

Here Ω is some domain in the real three-dimensional universe, $\partial\,\Omega$ is its boundary, n_1, n_2, n_3 are covariant components of the unit normal vector $\mathbf{n}$ perpendicular to the boundary $\partial\,\Omega$, and dS is the infinitesimal area element of this boundary. The integral equation (5.28) expresses the rest mass conservation law which is formulated in the following theorem.

Theorem 5.1. *The increment of the total rest mass of an elastic medium per unit time in a closed 3D-domain Ω is equal to the rest mass supplied to the domain per unit time through its boundary $\partial\,\Omega$.*

The equation (5.27) is a purely kinematic equation. It is derived without using any dynamic properties of an elastic medium.

6. Lagrangian of a relativistic elastic solid medium.

The elastic energy of a deformed elastic solid medium is written as the integral of its density in the real universe:

$$F=\int\mathcal{F}\,\sqrt{\det g}\;d^3x. \tag{6.1}$$

The density of the elastic energy $\mathcal{F}$ in (6.1) depends on the properties of a medium. These properties are initially referred to points of the static space with the coordinates y^1, y^2, y^3. They are transferred to the real universe by means of the immersion mapping (3.2), which is written as (5.7), and by means of the Jacobian matrix J with the components (5.5). The immersion (5.7) produces deformation of a medium which is the main source of the elastic energy. Therefore $\mathcal{F}$ depends on the deformation tensor $\mathbf{u}$ with the components (3.11). For a relativistic medium it can also depend on the velocity vector $\mathbf{v}$ with the components (3.5) directly. As a result we write the following formula for $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}(y, J, \mathbf{v}, \mathbf{u}). \tag{6.2}$$

Here y symbolizes a point of the static space with the coordinates y^1, y^2, y^3.

The action of the elastic solid matter is the time integral of its Lagrangian, while its Lagrangian is the spacial integral of its Lagrangian density:

$$S_{\text{mat}} = \int L_{\text{mat}}\,dt, \qquad\qquad L_{\text{mat}} = \int \mathcal{L}_{\text{mat}}\sqrt{\det g}\,d^3x. \tag{6.3}$$

Using (6.1) and (6.2), we write the Lagrangian density $\mathcal{L}_{\text{mat}}$ as follows:

$$\mathcal{L}_{\text{mat}} = -\rho\,c_{\text{br}}^2\sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c_{\text{br}}^2}} - \mathcal{F}(y, J, \mathbf{v}, \mathbf{u}). \tag{6.4}$$

Note that in (6.4) we see the third speed constant c_{br} from (2.2). Due to (2.4) it coincides with the second speed constant c_{gr} from (2.2), which is used in (5.27). However, if we consider a non-baryonic elastic medium, i. e. made of dark matter, then the constant c_{br} should be replaced by the fourth constant c_{nb} from (2.2).

7. Differential equations for the dynamics of a relativistic solid medium.

Differential equations in Lagrangian theories are derived by applying the stationary action principle to action integrals (see [40]). For the dynamic variables in (6.3) and (6.4) we choose the functions from (3.3) and their time derivatives

$$w^i = \dot{y}^i = \frac{\partial y^i(t, x^1, x^2, x^3)}{\partial t}, \quad 1 \leqslant i \leqslant 3. \tag{7.1}$$

In terms of the variables y^1, y^2, y^3, w^1, w^2, w^3 the Euler-Lagrange equations (see [41]) associated with the action integral (6.3) are written as follows:

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta w^i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},y}} - c_{\text{gr}}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta w^i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},y}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta y^i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},w}} = 0. \tag{7.2}$$

The equations (7.2) constitute a version of the equations (3.10) from [14]. The next step is to write the equations (7.2) in a more explicit form.

The components of the velocity vector (3.5) are used in (6.4). They are expressed through the functions (7.1) by means of the following formula:

$$v^i = -\sum_{r=1}^{3} M^i_r\,w^r. \tag{7.3}$$

The formula (7.3) is easily derived with the use of (5.7), (5.8), and (5.13). Apart from the velocity vector $\mathbf{v}$, in (6.4) we see the rest mass density (5.2). This function is determined by the formula (5.6). The constituent parts of the right hand side of (5.6) depend on the functions y^1, y^2, y^3 and their spacial derivatives. But they do not depend on the components of the velocity vector $\mathbf{v}$. Therefore

$$\frac{\partial\rho}{\partial v^r}=0. \tag{7.4}$$

Let's denote through $\mathcal{K}$ the first summand in the right hand side of (6.4). Then

$$\begin{aligned} &K=\int\mathcal{K}\,\sqrt{\det g}\;d^3x, && F=\int\mathcal{F}\,\sqrt{\det g}\;d^3x,\\ &\mathcal{K}=-\rho\,c_{\text{br}}^2\,\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}, && \mathcal{L}_{\text{mat}}=\mathcal{K}-\mathcal{F}. \end{aligned} \tag{7.5}$$

The expression for $\mathcal{K}$ in (7.5) has no spacial derivatives of the functions (3.5). Therefore its variational derivatives with respect to w^i coincide with partial derivatives. Applying (7.3), we derive the following formula:

$$\Bigl(\frac{\delta\mathcal{K}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}=\frac{\partial\mathcal{K}}{\partial w^i}=-\sum_{r=1}^{3}\frac{\partial\mathcal{K}}{\partial v^r}\,M_i^r. \tag{7.6}$$

Then, applying (7.5) to (7.6) and taking into account (7.4), we get

$$\Bigl(\frac{\delta\mathcal{K}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}=-\sum_{r=1}^{3}\frac{\rho\,v_r\,M_i^r}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,. \tag{7.7}$$

The next step is to calculate the variational derivative of $\mathcal{K}$ with respect to the dynamic variables y^1, y^2, y^3. For this purpose we consider their small variations

$$\hat y^i=y^i(t,x^1,x^2,x^3)+\varepsilon\,h^i(t,x^1,x^2,x^3), \tag{7.8}$$

where $\varepsilon\to 0$ and $h^i(t,x^1,x^2,x^3)$ are arbitrary smooth functions with compact support (see [42]). The variational derivative of $\mathcal{K}$ is defined through the formula

$$\hat K=K+\varepsilon\int\sum_{i=1}^{3}\Bigl(\frac{\delta\mathcal{K}}{\delta y^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}h^i\,\sqrt{\det g}\;d^3x+\ldots\,. \tag{7.9}$$

Note that the components of the velocity vector $\mathbf{v}$ in (7.3) depend on y^1, y^2, y^3 through M_r^i. Despite the equality (7.1) the quantities w^1, w^2, w^3 are treated as independent variables not sensitive to the variations (7.8). Therefore, applying (7.8) to (7.3), we get the following relationship:

$$\hat v^i=-\sum_{r=1}^{3}\hat M_r^i\,w^r. \tag{7.10}$$

Now let's recall that the Jacobian matrix M is inverse to the Jacobian matrix J with the components (5.5). Therefore the matrix $\hat M$ in (7.10) is obtained by substituting

$\hat{y}^1$, $\hat{y}^2$, $\hat{y}^3$ for y^1, y^2, y^3 in the inverse matrix J^{-1}. This yields

$$\hat{M}^i_r = M^i_r - \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} M^i_m \, \frac{\partial h^m}{\partial x^n} \, M^n_r + \ldots \, . \tag{7.11}$$

Note that y^1, y^2, y^3 are coordinates in the static space, not in the real universe. Therefore the upper index i in (7.8) and the upper index m in (7.11) both are not tensorial indices. Hence we can write (7.11) as follows:

$$\hat{M}^i_r = M^i_r - \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} M^i_m \, \nabla_n h^m \, M^n_r + \ldots \, . \tag{7.12}$$

A similar expression is available for the Jacobian matrix J in (5.5):

$$\hat{J}^i_j = J^i_j + \varepsilon \, \nabla_j h^i + \ldots \, . \tag{7.13}$$

Applying (7.12) to the formula (7.10), we derive

$$\hat{v}^i = v^i - \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} M^i_m \, \nabla_n h^m \, v^n + \ldots \, . \tag{7.14}$$

Now let's proceed to the rest mass density in (5.6). Applying (5.22) to (5.6), we derive the following formula for the density ρ:

$$\rho = \frac{\rho_0 \sqrt{\det \eta}}{\sqrt{\det g}} \, |\det J|. \tag{7.15}$$

From (7.13), applying Jacobi's formula for differentiating determinants (see [39]), we derive a formula for the determinant $\det \hat{J}$:

$$\det \hat{J} = \det J \left(1 + \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} M^n_m \, \nabla_n h^m \right) + \ldots \, . \tag{7.16}$$

Due to (2.8) the denominator of the fraction in (7.15) does not depend on y^1, y^2, y^3. It remains unchanged when applying (7.8) to (7.15). Due to (3.1) and (5.1) the numerator of the fraction in (7.15) is a function of y^1, y^2, y^3. Therefore

$$\hat{\rho}_0 \sqrt{\det \hat{\eta}} = \rho_0 \sqrt{\det \eta} + \varepsilon \sum_{m=1}^{3} \frac{\partial(\rho_0 \sqrt{\det \eta})}{\partial y^m} \, h^m + \ldots \, . \tag{7.17}$$

Due to (5.7) we can interpret ρ_0 and $\det \eta$ in (7.17) as functions of x^1, x^2, x^3 and t. Therefore we can write the formula (7.17) as

$$\hat{\rho}_0 \sqrt{\det \hat{\eta}} = \rho_0 \sqrt{\det \eta} + \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} \nabla_n (\rho_0 \sqrt{\det \eta}) \, M^n_m \, h^m + \ldots \, . \tag{7.18}$$

It is easy to see that the right hand sides of the formulas (7.16) and (7.18) are rather similar. We are going to combine these formulas by multiplying their left hand sides. Upon doing it we obtain the following formula:

$$\begin{gathered}\hat{\rho}_0\,\sqrt{\det\hat{\eta}}\,|\det\hat{J}|=\rho_0\,\sqrt{\det\eta}\,|\det J|\,\cdot\\ \cdot\left(1+\varepsilon\sum_{m=1}^{3}\sum_{n=1}^{3}M^n_m\,\frac{\nabla_n(\rho_0\,\sqrt{\det\eta}\,h^m)}{\rho_0\,\sqrt{\det\eta}}\right)+\ldots\,.\end{gathered}\tag{7.19}$$

Then we divide both sides of the equality (7.19) by $\sqrt{\det g}$ and take into account that $\det\hat{g}=\det g$. As a result we derive the formula

$$\hat{\rho}=\rho\left(1+\varepsilon\sum_{m=1}^{3}\sum_{n=1}^{3}M^n_m\,\frac{\nabla_n(\rho_0\,\sqrt{\det\eta}\,h^m)}{\rho_0\,\sqrt{\det\eta}}\right)+\ldots\,.\tag{7.20}$$

Due to (7.15) the formula (7.20) can be rewritten as

$$\hat{\rho}=\rho\left(1+\varepsilon\sum_{m=1}^{3}\sum_{n=1}^{3}M^n_m\,\frac{\nabla_n(h^m\,\rho\,\sqrt{\det g}/|\det J|)}{\rho\,\sqrt{\det g}/|\det J|}\right)+\ldots\,.\tag{7.21}$$

Now we proceed to the square root factor in the third formula (7.5). Applying (7.14) to this square root factor, we derive

$$\begin{gathered}c^2_{\text{br}}\,\sqrt{g_{00}-\frac{|\hat{\mathbf{v}}|^2}{c^2_{\text{br}}}}=c^2_{\text{br}}\,\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c^2_{\text{br}}}}\,+\\ +\,\varepsilon\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{v_i\,M^i_m\,\nabla_nh^m\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c^2_{\text{br}}}}}+\ldots\,.\end{gathered}\tag{7.22}$$

Due to (7.21) and (7.22) the integral defining the variational derivative in the formula (7.9) subdivides into the sum of two integrals:

$$\int\sum_{i=1}^{3}\Bigl(\frac{\delta\mathcal{K}}{\delta y^i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},w}}h^i\,\sqrt{\det g}\;d^3x=I_1+I_2.\tag{7.23}$$

These two integrals I_1 and I_2 are written as follows:

$$\begin{gathered}I_1=-\int\sum_{i=1}^{3}\sum_{n=1}^{3}M^n_i\,\frac{\nabla_n(h^i\,\rho\,\sqrt{\det g}/|\det J|)}{\sqrt{\det g}/|\det J|}\,\cdot\\ \cdot\,c^2_{\text{br}}\,\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c^2_{\text{br}}}}\,\sqrt{\det g}\;d^3x,\end{gathered}\tag{7.24}$$

$$I_2=-\int\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\rho\,v_m\,M^m_i\,\nabla_nh^i\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c^2_{\text{br}}}}}\,\sqrt{\det g}\;d^3x.\tag{7.25}$$

Both integrals (7.24) and (7.25) comprise spacial derivatives of the functions h^i from (7.8). In order to remove these spacial derivatives both integrals I_1 and I_2

should be transformed by means of integration by parts:

$$I_1=\int\sum_{i=1}^{3}\sum_{n=1}^{3}\nabla_n\!\left(\frac{M^n_i\,c_{\text{br}}^2\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}{\sqrt{\det g}/|\det J|}\right)\cdot \rho\sqrt{\det g}/|\det J|\,h^i\sqrt{\det g}\,d^3x,\tag{7.26}$$

$$I_2=\int\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\nabla_n\!\left(\frac{\rho\,v_m\,M^m_i\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)h^i\sqrt{\det g}\,d^3x.\tag{7.27}$$

Now we can apply (7.26) and (7.27) to (7.23). This yields

$$\left(\frac{\delta\mathcal{K}}{\delta y^i}\right)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}=\sum_{n=1}^{3}\nabla_n\!\left(\frac{M^n_i\,c_{\text{br}}^2\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}{\sqrt{\det g}/|\det J|}\right)\rho\sqrt{\det g}/|\det J|+ +\sum_{m=1}^{3}\sum_{n=1}^{3}\nabla_n\!\left(\frac{\rho\,v_m\,M^m_i\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right).\tag{7.28}$$

Assume for a while that we have an elastic medium with zero elastic response. For such a medium $\mathcal{F}=0$ and the the equations (7.2) reduce to

$$-\frac{\partial}{\partial t}\left(\frac{\delta\mathcal{K}}{\delta w^i}\right)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}-c_{\text{gr}}\left(\frac{\delta\mathcal{K}}{\delta w^i}\right)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}\sum_{q=1}^{3}b^q_q+\left(\frac{\delta\mathcal{K}}{\delta y^i}\right)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}=0,\tag{7.29}$$

When calculating the time derivative in (7.29) due to (7.7) we shall have to calculate the time derivative of the Jacobian matrix (5.13):

$$\frac{\partial M^r_i}{\partial t}=-\sum_{m=1}^{3}\sum_{n=1}^{3}M^r_n\,\frac{\partial J^n_m}{\partial t}\,M^m_i.\tag{7.30}$$

The equality (7.30) follows from the equality $M=J^{-1}$. Applying the formulas (5.5) and (7.1) to the equality (7.30), we derive

$$\frac{\partial M^r_i}{\partial t}=-\sum_{m=1}^{3}\sum_{n=1}^{3}M^r_n\,\nabla_m w^n\,M^m_i.\tag{7.31}$$

The following formula expresses w^n through the components of the velocity vector:

$$w^n=-\sum_{q=1}^{3}J^n_q\,v^q.\tag{7.32}$$

The formula (7.32) is derived from (7.3). Applying it to (7.31), we get

$$\frac{\partial M^r_i}{\partial t}=\sum_{q=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}M^r_n\,\nabla_m J^n_q\,v^q\,M^m_i+\sum_{m=1}^{3}\nabla_m v^r\,M^m_i.\tag{7.33}$$

Note that $|\det J| = \pm\det J = (\pm 1)\cdot\det J$. The constant factor ± 1 can be removed from (7.28). Then the formula (7.28) can be written as

$$\begin{gathered}\left(\frac{\delta\mathcal{K}}{\delta y^i}\right)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},w}}=\sum_{n=1}^{3}\rho\,c_{\text{br}}^2\,\nabla_{\!n}\!\left(\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\right)M^n_i\,+\\ +\sum_{n=1}^{3}\rho\,c_{\text{br}}^2\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\left(\nabla_{\!n}M^n_i-\frac{1}{2}\,\frac{\nabla_{\!n}(\det g)}{\det g}\,M^n_i\,+\right.\\ \left.+\,\frac{\nabla_{\!n}(\det J)}{\det J}\,M^n_i\right)+\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\rho\,v_m\,v^n}{\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,\nabla_{\!n}M^m_i\,-\\ +\sum_{m=1}^{3}\sum_{n=1}^{3}\nabla_{\!n}\!\left(\frac{\rho\,v_m\,v^n}{\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)M^m_i.\end{gathered}\tag{7.34}$$

Now let's calculate some terms from (7.34). We start with $\nabla_{\!n}M^n_i$:

$$\begin{gathered}\nabla_{\!n}M^m_i=\frac{\partial M^m_i}{\partial x^n}+\sum_{s=1}^{3}\Gamma^m_{ns}\,M^s_i=-\sum_{r=1}^{3}\sum_{s=1}^{3}M^m_r\,\frac{\partial J^r_s}{\partial x^n}\,M^s_i\,+\\ +\sum_{s=1}^{3}\Gamma^m_{ns}\,M^s_i=-\sum_{r=1}^{3}\sum_{s=1}^{3}M^m_r\,\nabla_{\!n}J^r_s\,M^s_i-\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}M^m_r\,\cdot\\ \cdot\,\Gamma^q_{ns}\,J^r_q\,M^s_i+\sum_{s=1}^{3}\Gamma^m_{ns}\,M^s_i=-\sum_{r=1}^{3}\sum_{s=1}^{3}M^m_r\,\nabla_{\!n}J^r_s\,M^s_i.\end{gathered}\tag{7.35}$$

Here Γ^m_{ns} are connection components of the metric connection associated with the metric (2.8) in the real universe. The covariant derivatives in (7.33) and (7.35) are calculated with the use of these connection components. From (7.35) we derive

$$\nabla_{\!n}M^n_i=-\sum_{r=1}^{3}\sum_{s=1}^{3}M^n_r\,\nabla_{\!n}J^r_s\,M^s_i.\tag{7.36}$$

For the term with $\det g$ in (7.34) we get

$$\begin{gathered}\frac{\nabla_{\!n}(\det g)}{\det g}=\frac{1}{\det g}\,\frac{\partial(\det g)}{\partial x^n}=\sum_{r=1}^{3}\sum_{s=1}^{3}g^{rs}\,\frac{\partial g_{rs}}{\partial x^n}=\\ =\sum_{r=1}^{3}\sum_{s=1}^{3}g^{rs}\,\nabla_{\!n}g_{rs}+\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\Gamma^q_{nr}\,g_{qs}\,g^{rs}+\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\Gamma^q_{ns}\,g_{rq}\,g^{rs}.\end{gathered}\tag{7.37}$$

In (7.37) we used Jacobi's formula for differentiating determinants (see [39]). If we take into account $\nabla_{\!n}g_{rs}=0$ (see § 7 of Chapter III in [43]), then (7.37) reduces to

$$\frac{\nabla_{\!n}(\det g)}{\det g}=2\sum_{s=1}^{3}\Gamma^s_{ns}.\tag{7.38}$$

The next is the term with $\det J$ in (7.34). For this term we derive

$$\begin{gathered}\frac{\nabla_n(\det J)}{\det J}=\frac{1}{\det J}\,\frac{\partial(\det J)}{\partial x^n}=\sum_{r=1}^{3}\sum_{s=1}^{3}M^r_s\,\frac{\partial J^s_r}{\partial x^n}=\sum_{r=1}^{3}\sum_{s=1}^{3}M^r_s\,\nabla_n J^s_r\,+\\ +\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\Gamma^q_{nr}\,J^s_q\,M^r_s=\sum_{r=1}^{3}\sum_{s=1}^{3}M^r_s\,\nabla_n J^s_r+\sum_{r=1}^{3}\Gamma^r_{nr}.\end{gathered}\tag{7.39}$$

Applying (7.35), (7.36), (7.38), and (7.39) to (7.34), we get the relationship

$$\begin{gathered}\Bigl(\frac{\delta\mathcal{K}}{\delta y^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}=\sum_{n=1}^{3}\rho\,c_{\text{br}}^2\,\nabla_n\Biggl(\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\Biggr)\,M^n_i\,-\\ -\sum_{n=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\rho\,c_{\text{br}}^2\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\,\Bigl(M^s_r\,\nabla_s J^r_n\,M^n_i-M^s_r\,\nabla_n J^r_s\,M^n_i\Bigr)-\\ -\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\rho\,v_m\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,M^m_r\,\nabla_n J^r_s\,M^s_i\,+\\ +\sum_{m=1}^{3}\sum_{n=1}^{3}\nabla_n\Biggl(\frac{\rho\,v_m\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\Biggr)\,M^m_i.\end{gathered}\tag{7.40}$$

Now, using the symmetry of the connection components $\Gamma^q_{sn}=\Gamma^q_{ns}$, we derive

$$\nabla_s J^r_n=\frac{\partial^2 y^r}{\partial x^s\,\partial x^n}+\sum_{q=1}^{3}\Gamma^q_{sn}\,J^r_q=\frac{\partial^2 y^r}{\partial x^n\,\partial x^s}+\sum_{q=1}^{3}\Gamma^q_{ns}\,J^r_q=\nabla_n J^r_s.\tag{7.41}$$

In deriving (7.41) we used (5.5). Applying (7.41) to (7.40), we obtain

$$\begin{gathered}\Bigl(\frac{\delta\mathcal{K}}{\delta y^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}=\sum_{n=1}^{3}\rho\,c_{\text{br}}^2\,\nabla_n\Biggl(\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\Biggr)\,M^n_i\,-\\ -\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\rho\,v_m\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,M^m_r\,\nabla_n J^r_s\,M^s_i\,+\\ +\sum_{m=1}^{3}\sum_{n=1}^{3}\nabla_n\Biggl(\frac{\rho\,v_m\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\Biggr)\,M^m_i.\end{gathered}\tag{7.42}$$

Now we proceed to the time derivative in (7.29). Differentiating (7.7), we get

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{K}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}=\sum_{r=1}^{3}\frac{\partial}{\partial t}\Biggl(\frac{\rho\,v_r}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\Biggr)\,M^r_i+\sum_{r=1}^{3}\frac{\rho\,v_r}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,\frac{\partial M^r_i}{\partial t}.\tag{7.43}$$

Then we apply (7.33) to (7.43). This yields

$$
\begin{gathered}
-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{K}}{\delta w^i}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},y}}=\sum_{r=1}^{3}\frac{\partial}{\partial t}\left(\frac{\rho\,v_r}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)M^r_i\,+\\
+\sum_{r=1}^{3}\sum_{m=1}^{3}\frac{\rho\,v_r\,\nabla_{\!m}v^r\,M^m_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}+\sum_{r=1}^{3}\sum_{q=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\rho\,v_r\,M^r_n\,\nabla_{\!m}J^n_q\,v^q\,M^m_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}.
\end{gathered}\tag{7.44}
$$

Now we can substitute (7.44), (7.42), and (7.7) into the equation (7.29). In doing it we find that due to the symmetry (7.41) the last term in the right hand side of (7.44) and the second term in the right hand side of (7.42) do cancel each other. As a result we get the following equation:

$$
\begin{gathered}
\sum_{n=1}^{3}\frac{\partial}{\partial t}\left(\frac{\rho\,v_n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)M^n_i+\sum_{n=1}^{3}\sum_{m=1}^{3}\nabla_{\!m}\left(\frac{\rho\,v_n\,v^m}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)M^n_i\,+\\
+\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{\rho\,v_m\,\nabla_{\!n}v^m\,M^n_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}+\sum_{n=1}^{3}\rho\,c_{\text{br}}^2\,\nabla_{\!n}\left(\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\right)M^n_i\,+\\
+\sum_{n=1}^{3}\sum_{k=1}^{3}\frac{\rho\,c_{\text{gr}}\,v_n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,b^k_k\,M^n_i=0.
\end{gathered}\tag{7.45}
$$

Note that the components of the matrix M enter each term of (7.45) in the same way. This matrix is non-degenerate. Therefore we can remove it from (7.45) at all. Moreover, we can calculate the covariant derivative in the fourth term of (7.45):

$$
\nabla_{\!n}\left(\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\right)=\frac{1}{2}\,\frac{\nabla_{\!n}g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}-\frac{1}{c_{\text{br}}^2}\sum_{m=1}^{3}\frac{v_m\,\nabla_{\!n}v^m}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,.\tag{7.46}
$$

Applying (7.46) to (7.45) and removing the components of the matrix M, we get

$$
\begin{gathered}
\frac{\partial}{\partial t}\left(\frac{\rho\,v_n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)+\sum_{k=1}^{3}\frac{\rho\,c_{\text{gr}}\,v_n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,b^k_k\,+\\
+\sum_{m=1}^{3}\nabla_{\!m}\left(\frac{\rho\,v_n\,v^m}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\right)=-\frac{1}{2}\,\frac{\rho\,c_{\text{br}}^2\,\nabla_{\!n}g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,.
\end{gathered}\tag{7.47}
$$

The equation (7.47) is analogous to the equation (5.27): the first term of (7.47) is an analog of the first term in (5.27), the second term of (7.47) is an analog of the second term in (5.27), the third term of (7.47) is an analog of the third term in

(5.27). The term in the right hand side of (7.47) has no analogs in (5.27). However it has an analog in the right hand side of the equation (3.6) in [15].

The equation (7.47) is analogous to the equation (5.6) in [36]. This analogy can be strengthened if we introduce the following notations:

$$p_n=\frac{\rho\,v_n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,,\qquad\qquad \Pi_n^m=\frac{\rho\,v_n\,v^m}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,.\tag{7.48}$$

The quantities p_n in (7.48) are interpreted as the components of the momentum density covector. The quantities Π_n^m in (7.48) are interpreted as the components of the momentum flow density tensor. In terms of the quantities introduced in (7.48) the equation (7.47) is written as follows:

$$\frac{\partial p_n}{\partial t}+\sum_{k=1}^{3}c_{\text{gr}}\,p_n\,b_k^k+\sum_{m=1}^{3}\nabla_m\Pi_n^m=f_n.\tag{7.49}$$

The quantities f_n in (7.49) are given by the formula

$$f_n=-\frac{1}{2}\,\frac{\rho\,c_{\text{br}}^2\,\nabla_n g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,.\tag{7.50}$$

The quantities (7.50) are interpreted as the components of the bulk gravitational forces density vector arising due to the gradient of the gravitational field. The main result of this section is presented by the following theorem.

Theorem 7.1. *The dynamics of a relativistic elastic medium with zero elastic response is described by the equation* (7.49) *complemented with the equation* (5.27).

The above equation (7.49) is similar to the equation (5.2) in [36].

If the elastic response of a medium is nonzero, then we shall have one more term in the right hand side of the equation (7.49):

$$\frac{\partial p_n}{\partial t}+\sum_{k=1}^{3}c_{\text{gr}}\,p_n\,b_k^k+\sum_{m=1}^{3}\nabla_m\Pi_n^m=f_n+\mathcal{F}_n.\tag{7.51}$$

The quantities $\mathcal{F}_n$ in (7.51) are given by the formula

$$\mathcal{F}_n=\sum_{i=1}^{3}\biggl(-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{F}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}-c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{F}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}}\sum_{q=1}^{3}b_q^q+\Bigl(\frac{\delta\mathcal{F}}{\delta y^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}\biggr)\,J_n^i.\tag{7.52}$$

The formula (7.52) is derived from the Euler-Lagrange equation (7.2) by applying the last formula (7.5). Now we can formulate the following theorem.

Theorem 7.2. *The dynamics of a general relativistic elastic medium is described by the equation* (7.51) *complemented with the equation* (5.27).

Through $\mathcal{F}$ in (7.52) we denote the function (6.2). In this paper we do not consider any specific forms of this function. This will be done in a separate paper.

8. Concluding remarks.

The main results of the present paper are Theorem 5.1, Theorem 7.1, Theorem 7.2, and the formula (3.11) for the relativistic nonlinear deformation tensor that generalizes the classical formula (3.7). The formula (3.11) takes into account the relativistic contraction of solid bodies along the direction of their motion and reproduces this phenomenon in the framework of the 3D-brane universe model in the form of a deformation-free steady flow considered in section 4 above.

Note that throughout this paper we see the speed constant c_{br}. This means that we deal with regular baryonic matter. In the case of non-baryonic dark matter, provided this matter is able to form condensed media, we would have to replace c_{br} with the other constant c_{nb} from (2.2).

6. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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