SCIENTIFIC ELECTRONIC PUBLICATION

R. A. SHARIPOV

3D-BRANE UNIVERSE MODEL

Monograph, part I

UFA 2025

«Just an instance short as a wink is between the future and the past, this very instance is called life!» These are the words from the song for the movie «Sannikov Land». They could not be better suited as an epigraph to this book.

For a person this instance is his current thoughts and feelings. Or the actions that he is performing now. For the universe this instance is stretched across all its vast expanses and includes all the events that are happening now, no matter how far away from us they are. Such an instance can be imagined as a three-dimensional film or membrane, which for brevity is called a 3D-brane. It separates the four-dimensional bulk of the past from the four-dimensional bulk of the future.

In his theory of relativity Albert Einstein combined space and time into one four-dimensional continuum. He forbade drawing a boundary between the past and the future in such a continuum, calling it conditional and dependent on the observer. According to the author of this e-book, now it is time to return to the ideas of Isaac Newton and draw a boundary between the past and the future. But now, at a new stage of development, this boundary is no longer flat, but it is flexible. It can bend, which manifests itself through gravitational lensing. This boundary can also stretch, which manifests itself in the form of the expansion of the universe and in the scattering of distant galaxies in all directions from us.

As a bonus in the new theory, which is presented in this book, there is the opportunity to move faster than the speed of light. It is realized in dark matter particles which are called superbradyons. These particles were invented by Luis Gonzalez-Mestres. He also gave them this name — superbradyons.

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In preparing the Russian edition of this book I used computer typesetting based of the $\mathcal{AMS}$-TEX package and I used Cyrillic fonts of the Lh-family distributed by CyrTUG association of Cyrillic TEX users. The English edition of this book is also typeset by means of the $\mathcal{AMS}$-TEX package. In preparing the English edition of the book the assistance of the Google Translate service was used.

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Книга является первой из серии книг, в которых излагается новая неэйнштейновская теория гравитации, получившая название «Модель вселенной как 3D-браны». Книга адресована студентам, аспирантам и научным работникам, специализирующимся в области математической физики, астрофизики, космологии, астрономии и физики элементарних частиц.

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CONTENTS.

PREFACE.

In the history of physics there were periods of steady accumulation and recognizing the knowledge about nature which alternated with periods of revolutionary changes of our views of nature. One of such periods of revolutionary changes is associated with the emergence of Albert Einstein's theory of relativity and with the emergence of quantum mechanics which was authored by several scientists. As to me, unlike the opinion of many others, this revolutionary period is not yet finished and we can expect some corrections to the picture of the world that was drawn by the theory of relativity and quantum mechanics.

The 3D-brane universe model is a new non-Einsteinian theory of gravity that is based on criticism and some adjustment of the concept of spacetime. In this book we present the first part of this theory covering the period of its development from the summer of 2022 to the spring of 2024. It includes

— an exposition of arguments in favor of the need to make changes to the theory of relativity;
— a formulation of the basic concepts of the new theory;
— a derivation of the equations of gravity in the new theory;
— a derivation of the total energy conservation law in the new theory;
— a derivation of formulas for the density of energy of a gravitational field and for the density of energy flux of a gravitational field in the new theory;
— a description of the motion of classical (non-quantum) matter particles in a gravitational field within the framework of the new theory.

The content of further parts of the new theory will be determined over time as it further develops.

August 2024. R. A. Sharipov.

CHAPTER I

BASIC CONCEPTS AND STRUCTURES.

§ 1. Criticism of spacetime.

Spacetime is a four-dimensional continuum that was constructed by joining three-dimensional space and one-dimensional time. It is in the basis of both special and general relativity (see [1–3]). In special relativity it is a flat continuum. In general relativity this continuum is endowed with a curvature determined by the gravitational field in it.

Points of spacetime are called events, while spacetime itself comprises all events that have happened anywhere and at any time. This means that it comprises the past, the present, and the future. The boundary between the past and the future in the theory of relativity is fuzzy, it depends on the observer. This circumstance is called the relativity of simultaneity. It leads to a paradox which is called the Andromeda paradox (see [4] and [5], pp. 303-304). In a slightly different formulation it is known as the Napoleon paradox (see [6]).

In philosophical literature the Andromeda paradox is known as the Rietdijk-Putnam argument (see [7] and [8]). It is considered an argument for the four-dimensionality of the physical universe. However, in reality it demonstrates a contradiction between four-dimensionality and common sense.

Instead of considering the entire universe, we can limit ourselves to some part of it. For example to the Earth Globe. Then

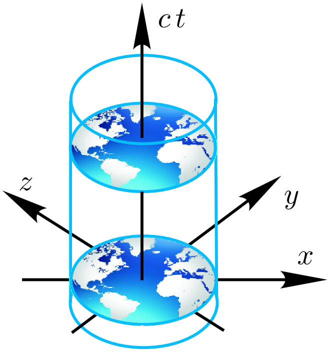

Fig. 1.1

we will get a cylindrical structure shown in Fig. 1.1. Such a cylindrical structure resembles a sausage in shape. Therefore we can call it a spacetime sausage. This sausage contains the entire history of the planet Earth, which includes the period of the gas-dust cloud and the primary Earth, the period of the appearance of liquid water and the origin of life, the era of dinosaurs, the era of mammals, the appearance of Homo sapiens, our present, i. e. the current moment in time, and our entire future.

The main question addressed to the theory of relativity and related to spacetime is formulated as follows.

QUESTION 1.1. *Is four-dimensional spacetime a physical continuum? Or is it just a product of our minds — a mathematical abstraction to which nothing corresponds in reality?*

This question is discussed among philosophers (see [9]). Physicists avoid it, relying on the authority of Einstein's theory of relativity. In this theory spacetime is considered a physical continuum by default. Indeed, in it the equations of gravity are written in a four-dimensional formalism, Maxwell's equations of electrodynamics are rewritten in a four-dimensional form, and the equations of motion of material bodies and various material media also tend to be brought to a four-dimensional form.

The choice in favor of the conviction that spacetime is a physical continuum is a responsible decision. It follows from this choice that the spacetime sausage containing the entire history of the Earth is a physical object. It contains in its original form the gas-dust cloud and the primary hot Earth, the first oceans with the first bacteria, dinosaurs, and mammoths with saber-toothed tigers. Moreover, the spacetime sausage contains our entire future, which has not yet arrived.

Since the past does not disappear, being preserved in the spacetime sausage, and since the future is predetermined and already formed in the spacetime sausage, Einstein's theory of relativity admits the potential possibility to travel back to the past and forward to the future. Although the mechanism for such travels is not spelled out, they are very popular in the genre of science fiction. **But neither in Einstein's time nor now have there been or are there any experimental demonstrations of time travel. Therefore, the position that spacetime is a physical continuum is an unproven and controversial point in the foundations of Einstein's theory of relativity.**

§ 2. Three-dimensional universe and its presentation in the form of 3D-branes.

Returning to the main question 1.1, we emphasize that, unlike the theory of relativity, the answer to it in the new theory is negative. This means that spacetime is not a physical continuum. However, we do not completely abandon the concept of four-dimensional spacetime and use it as a valuable mathematical tool for selectively transferring some individual results from the theory of relativity to the new theory.

Spacetime in the theory of relativity is a four-dimensional manifold equipped with three geometric structures: 1) a pseudo-Riemannian metric with the signature $(+,-.-,-)$, 2) an orientation, 3) a polarization (see [3]). An observer in the theory of relativity is an animate object whose dimensions are small compared to planets, stars, and galaxies so that it can be considered as a point object. The motion of observers in spacetime is depicted in the form of their world lines. Figure 2.1 (see below) shows the world lines of two observers. At each point they pass inside light cones determined by the spacetime metric.

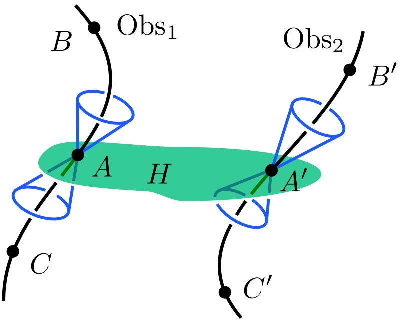

Fig. 2.1

The motion of observers along their world lines goes from the past through the present to the future, which is determined by the polarization of spacetime. In our case, this is the motion from bottom to top.

Let us consider separately the first observer in Fig. 2.1. Let the point A be the point of the present for the first observer. It has material existence, i. e. it has a prototype in the real physical universe. The point C is the point of the past for the first observer. It had, but lost its material existence, remaining in the past. The point B, the point of the future for the first observer, has not yet acquired its material existence. Thus, on each world line at each moment only one point has material existence. This property (material existence) is transient. It passes from one point to another as events unfold in the real physical universe.

Let us consider again the first observer in Fig. 2.1, who is at the point A in his present. Being at this point, he understands that at this moment he is not alone in the universe. Somewhere there is some second observer, who at this moment is at some point A' on his world line. We know that instantaneous data transmission is forbidden in the theory of relativity. It is forbidden in the new theory as well. The points A and A' are not connected to each other by any signals. They have only one thing in common — **joint material existence**. Joint material existence between points A and A' exists in the present. But such a connection between points of two world lines could have existed in the past and can be formed in the future. For example, the point C on the world line of the first observer could have joint material existence with some point C' on the world line of the second observer if the second observer existed at that moment,

i. e. if he was born, but has not yet died. Similarly, the point B on the world line of the first observer will have joint material existence with some point B' on the world line of the second observer if the second observer will exist at that moment, i. e. if he will be born, but will not have died yet.

In what follows we consider the property of **joint material existence** as a binary equivalence relation (see [10] or § 2 in Chapter I of [11]) on spacetime regardless of when it occurs — in the past, in the present, or in the future. We do not restrict it to points on world lines of animate observers and extend it to points on world lines of inanimate objects, as well as to vacuum points in interstellar space and to vacuum points inside vacuum chambers of man-made apparata. The binary relation of **joint material existence** partitions spacetime into pairwise disjoint **classes of joint material existence**. One of such classes, namely the class of joint material existence of the points A and A', is depicted in Fig. 2.1 in the form of a colored spot.

Speculatively the form and structure of classes of joint material existence can be any. They can be smooth structures, or fractals, or even completely structureless sets. But we prefer to deal with smooth structures and postulate that they are smooth orientable three-dimensional manifolds, i. e. 3D-branes.

The 3D-branes of classes of joint material existence are subject to the natural requirement of spacelikeness. It means that at all their points the tangent hyperplanes to these branes intersect the corresponding light cones only at their vertices.

From the above the following picture of the world is formed in the new theory. The real physical universe is three-dimensional. It evolves and each moment of its evolution is depicted in the form of a 3D-brane in four-dimensional spacetime. These 3D-branes fill the entire spacetime with the possible exception of one point, which corresponds to the Big Bang (see [12]). Thus, the new theory of gravity considered in this book denies the material existence of the entire spacetime as a whole and turns it into a collection of mathematical images of the real physical universe obtained at different moments of its evolution. It endows the spacetime of the theory of relativity with one more geometric structure — a foliation of spacelike 3D-branes. In what follows we shall consider spacetime as a four-dimensional manifold equipped with four geometric structures: 1) a pseudo-Riemannian metric with the signature $(+,-.-,-)$, 2) an orientation, 3) a polarization, 4) a foliation of spacelike 3D-branes filling it entirely with the exception of perhaps one point corresponding to the Big Bang. Spacetime with such structures serves as a factor of continuity and a bridge between the theory of relativity and the new theory.

§ 3. The field of unit normals and comoving spacial coordinates.

According to the results of the previous section, spacetime is now equipped with a foliation of spacelike 3D-branes. The 3D-branes fill the entire spacetime except for perhaps one point corresponding to the Big Bang. If we exclude this point, then through each of the remaining points P there passes exactly one 3D-brane. The spacelikeness of the branes means that the perpendiculars to them are timelike. The orientability of the branes and the presence of a metric and polarization in spacetime allow us to choose unit normal vectors to the branes $\mathbf{n}(P)$ directed toward the future and changing smoothly when moving from point to point within individual branes and when moving from one brane to another:

$$|\mathbf{n}(P)| = 1. \tag{3.1}$$

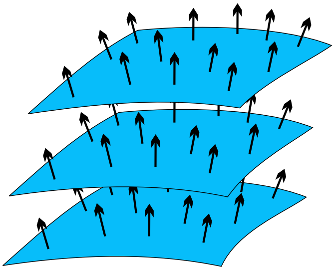

Fig. 3.1

The unit normal vectors (3.1) to the branes constitute a smooth vector field in the foliation of 3D-branes. This vector field is shown in Fig. 3.1. We cannot draw three-dimensional branes in a four-dimensional space, therefore in the figure they are drawn in the form of two-dimensional branes in a three-dimensional space.

Every vector field has a family of field lines associated with it (see [13]). These are lines whose tangent vector at each of their points is directed along the field vector at that point. Field lines of unit normals **n** are shown in Fig. 3.2.

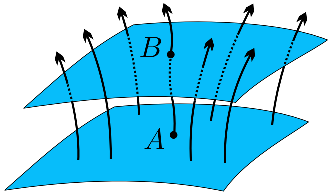

Fig. 3.2

The concept of a field line is very similar to the concept of an integral line for a vector field (see [14]). The difference is in the parameterization. Integral lines of a vector field are parametric lines whose tangent vector in their parameterization at each of their points coincides with the field vector at that point. Geometrically, as sets of points, integral lines of a vector field coincide with its field lines. Therefore in what follows we shall not make a distinction between the field lines and the integral lines of the vector field **n** in Fig. 3.2.

Let us choose some arbitrary curvilinear coordinates x, y, z on one of the 3D-branes in Fig. 3.2, say on the lower one, and introduce the notation:

$$x^1 = x, \qquad x^2 = y, \qquad x^3 = z. \tag{3.2}$$

The use of superscripts for numbering coordinates of vectors and some other indexing conventions are believed to have been invented by Einstein. They constitute Einstein's tensorial notation for the use of indices (see § 20 in Chapter I of [15]).

Using the field lines of the unit normal vector field **n** (see Fig. 3.2), the coordinates (3.2) can be extended from an initially chosen 3D-brane to all other branes, both upwards to the future and downwards to the past. The coordinates obtained in this way are called comoving coordinates.

Definition 3.1. Three smooth functions x, y, z defined globally in the entire spacetime or locally in some region of it are called comoving spacial coordinates if their values do not change when moving along field lines of the unit vector field perpendicular to the 3D-branes and if they become global or local coordinates on a brane after restriction to any of the 3D-branes.

The choice of comoving coordinates is not unique. We can replace the initially chosen comoving coordinates by others. Such a replacement of comoving coordinates is carried out within some individual 3D-brane and then it extends to all other branes. Therefore we obtain the following formulas for the transition from some initially chosen comoving spacial coordinates to any other comoving spacial coordinates:

$$\begin{cases} \tilde{x}^1 = \tilde{x}^1(x^1, x^2, x^3), \\ \tilde{x}^2 = \tilde{x}^2(x^1, x^2, x^3), \\ \tilde{x}^3 = \tilde{x}^3(x^1, x^2, x^3), \end{cases} \qquad \begin{cases} x^1 = x^1(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \\ x^2 = x^2(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), \\ x^3 = x^3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3). \end{cases} \tag{3.3}$$

Spacetime is four-dimensional. But in (3.3) we see only three coordinates. The fourth coordinate does not participate in the replacement of comoving coordinates.

§ 4. Comoving observers and the state of absolute rest.

Let's recall that 3D-branes, which constitute the new fourth geometric structure in spacetime, are spacelike hypersurfaces. The unit vectors normal to them are timelike vectors. This means that the field lines of the unit vectors normal to the branes, shown in Fig. 3.2, can serve as worldlines of some observers in spacetime. Such observers are called comoving observers.

Definition 4.1. Observers whose comoving coordinates do not change over time are called comoving observers.

Comoving observers move perpendicular to the 3D-branes from the past to the future. They do not move in the direction along the branes. Therefore, they are considered to be at rest. This is the state of absolute rest, since it is not tied to any material objects in the universe. Comoving coordinates form a dedicated coordinate systems in the universe that define the state of absolute rest.

QUESTION 4.1. *Is the presence of dedicated coordinates defining the state of absolute rest a necessary condition in the new theory?*

The answer to this question is negative. Dedicated coordinates appear in the new theory because we do not completely reject the legacy of Einstein's theory of relativity and retain the concept of spacetime, though we lower its status to the level of an immaterial mathematical abstraction. In principle, it is possible to construct a theory of a three-dimensional universe without using the concept of spacetime. In such a theory, there may be no dedicated coordinate systems and no state of absolute rest.

QUESTION 4.2. *Is the presence of dedicated coordinates that define the state of absolute rest a return to the ether theory?*

Yes, to some extent. Although the classical luminiferous ether of the 19th century is the medium in which light propagates. In our case, comoving coordinates and the state of absolute rest determined by them are not related to any material medium.

QUESTION 4.3. *Is the presence of dedicated coordinates defining the state of absolute rest a return to absolute Newtonian three-dimensional space?*

Yes, to some extent. But Newtonian three-dimensional space is flat and unchanging. In our case, the metric on different 3D-branes can be different and non-flat. This means that the metric in our three-dimensional universe can be non-flat and change over time. That is, the universe in our theory can expand or contract in some individual regions or globally as a whole.

§ 5. Membrane time.

Definition 5.1. A smooth numerical function t on spacetime is called membrane time if its values do not change within each 3D-brane of the foliation of 3D-branes and if it increases strictly monotonically in the direction from the past to the future.

We know that 3D branes correspond to different stages in the evolution of the real three-dimensional universe. Membrane time numbers these stages assigning each of them some numerical value from the set of real numbers.

The choice of membrane time is not unique. The replacement of one membrane time by another is given by the formulas

$$\tilde{t} = \tilde{t}(t), \qquad\qquad t = t(\tilde{t}). \tag{5.1}$$

The transformations (5.1) are called membrane time scaling transformations. Smooth functions of one variable in (5.1) are subject to additional conditions

$$\frac{d\tilde{t}}{dt} > 0, \qquad\qquad \frac{dt}{d\tilde{t}} > 0. \tag{5.2}$$

The conditions (5.2) ensure strict monotonicity of the functions $\tilde{t}(t)$ and $t(\tilde{t})$ in the formulas (5.1).

When applied to the real three-dimensional physical universe, membrane time is a global time, it is defined throughout the universe and is the same at all its points. But, being simply a marker numbering 3D-branes and distinguishing them from each other in the foliation of 3D-branes, membrane time does not have to coincide with the time measured by any device.

§ 6. The equidistance postulate and abandonment of it.

The first version of the new theory of gravity, the name of which coincides with the title of this book, was developed in a series of publications [16–21]. The works [16–21] were preceded by the work [22]. The results of the works [16–21] were reported at the conferences [23–27]. The first version of the theory was constructed using the following equidistance postulate.

POSTULATE 6.1. *For any two 3D-branes from the foliation of 3D-branes in spacetime the lengths of all segments of the field lines in Fig. 3.2 enclosed between these two 3D-branes are the same.*

Later I realized that the equidistance postulate 6.1 is not needed. In the second version of the theory it was excluded, see the works [28–33] and the conference abstracts [34–37]. The second version of the theory without the equidistance postulate is more general. Therefore it is presented further in this book.

CHAPTER II

GRAVITATIONAL FIELD EQUATIONS.

§ 1. Speed of light and its analogs.

The speed of light in vacuum is the speed of propagation of electromagnetic waves in empty space. Accordingly we shall denote it by c_{el}. Generally speaking this is an experimentally measurable quantity. However, in 1983 by resolution No. 1 adopted at the 17th meeting the General Conference on Weights and Measures decided to define the standard of length of 1 meter through the speed of light in vacuum. After that the quantity c_{el} received an exact numerical value

$$c_{\text{el}} = 299792458 \text{ m/s} \tag{1.1}$$

(see [38]). In addition to the unit of length, the formula (1.1) uses a unit of time. This is the second. Since 1967, one second has been defined as 9192631770 periods of oscillations of the radiation corresponding to the transition between two levels of the hyperfine structure in the ground state of the cesium-133 isotope atom (see [39] and [40]).

In Einstein's theory of relativity the speed of light plays many roles. In addition to determining the speed of propagation of electromagnetic waves it is present in the equations of the gravitational field and it determines the speed limit of motion of massive material bodies. The material bodies that we observe in everyday life consist of matter that in astrophysics is called light or baryonic matter. In addition to it, there is so-called dark matter (see [41]). It is not detected by direct observations and experiments. Its presence is confirmed indirectly by determining the speeds of stars on the outskirts of galaxies (see [42]) and through gravitational lensing (see [43]). Since there is currently no way to experimentally measure the maximum speed for dark matter, there is no reason to believe that this speed coincides with the constant (1.1). In this book we shall consider four speed constants. They are

$$c_{\text{el}}, \qquad c_{\text{gr}}, \qquad c_{\text{br}}, \qquad c_{\text{nb}}. \tag{1.2}$$

The first constant (1.2) coincides with the constant (1.1). The second is used in the gravity equations. The third is the limiting speed for baryonic matter. The fourth constant is the limiting speed for non-baryonic matter. Since we currently know nothing about the structure of dark matter, we assume that it can be divided into several sorts and each sort of dark matter can have its own value of the constant c_{nb}.

In the new theory of gravity, which is considered in this book, there are no a priori prohibitions on all the constants (1.2) being different. And if an experiment shows that some of them coincide, then this must be given a separate theoretical justification. We do not consider Einstein's theory of relativity to be such a justification due to the objections to it that were expressed in § 1 from the first chapter of this book.

§ 2. Reduction of a four-dimensional metric to a three-dimensional one.

Based on the criticism of spacetime in § 1 of the first chapter, we have reduced its status to the level of a mathematical abstraction, to which no real four-dimensional physical space corresponds. However, in the new theory we do not abandon the concept of spacetime completely, retaining it as a useful mathematical abstraction.

In § 2 of Chapter 1, it was said that spacetime is equipped with four geometric structures: 1) a pseudo-Riemannian metric with the signature $(+,-.-,-)$, 2) an orientation, 3) a polarization, 4) a foliation of spacelike 3D-branes filling it entirely except for perhaps one point corresponding to the Big Bang. The first three of these structures are borrowed from Einstein's theory of relativity. The fourth is added in the new theory based on the arguments in § 2 of Chapter 1. Let us examine the role of these structures. The orientation prevents left and right from mixing in dimension four and prevents spacetime from being something bad like a Möbius strip (see [44]).

The polarization indicates the direction from the past to the future. In § 3 of the first chapter, it allowed us to choose a field of unit normals to the 3D-branes directed to the future. The presence of the first three structures and the spacelikeness of the 3D-branes induces a three-dimensional orientation to them. That is, in the 3D-branes the left and right in dimension three also cannot mix and the 3D-branes themselves cannot be something bad like a Möbius strip (see [44]). This is natural since 3D-branes in our theory are images of the real physical universe at different moments of its evolution.

The pseudo-Riemannian metric is the basic quantitative characteristic of spacetime. In an arbitrary coordinate system x^0, x^1, x^2, x^3 it is defined by a symmetric 4×4 matrix G. The components of this matrix are

$$G_{ij} = G_{j\,i}, \text{ where } 0 \leqslant i, j \leqslant 3. \tag{2.1}$$

In § 3 of Chapter I we constructed three special spacial coordinates (3.2) associated with the foliation of 3D-branes and the field of unit normals $\mathbf{n}$ to them. They were called comoving coordinates, see definition 3.1. Then, in § 5 of Chapter I we defined the membrane time t, see definition 5.1. Using the membrane time, we complement the spacial comoving coordinates (3.2) to a complete coordinate system in four-dimensional spacetime:

$$x^0 = c_{\text{gr}}\,t, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z. \tag{2.2}$$

Note that in (2.2) we do not use the speed of light c_{el} from the formula (1.1), but we use the second constant from (1.2).

With the coordinates (2.2) the vectors

$$\mathbf{e}_0 = \frac{\partial}{\partial x^0}, \qquad \mathbf{e}_1 = \frac{\partial}{\partial x^1}, \qquad \mathbf{e}_2 = \frac{\partial}{\partial x^2}, \qquad \mathbf{e}_3 = \frac{\partial}{\partial x^3} \tag{2.3}$$

are associated. From the definition 5.1 in the first chapter it follows that individual 3D-branes can be distinguished by conditions of the form $t = \text{const}$. Therefore the last three vectors in (2.3) are tangent to 3D-branes. From the definition 3.1 in the first chapter it follows that individual field lines of the vector field $\mathbf{n}$ can be distinguished by conditions of the form $x^1 = \text{const}_1$, $x^2 = \text{const}_2$, $x^3 = \text{const}_3$. Therefore the vector $\mathbf{e}_0$ is tangent to the field lines of the field $\mathbf{n}$ and is directed to the future along the normal vector $\mathbf{n}$. Hence

$$\mathbf{e}_0 \perp \mathbf{e}_1, \qquad \mathbf{e}_0 \perp \mathbf{e}_2, \qquad \mathbf{e}_0 \perp \mathbf{e}_3. \tag{2.4}$$

For the components of the pseudo-Riemannian metric (2.1) in the coordinates (2.2) the relationships (2.4) mean that

$$\begin{aligned} &G_{12} = 0, && G_{13} = 0, && G_{23} = 0,\\ &G_{21} = 0, && G_{31} = 0, && G_{32} = 0. \end{aligned} \tag{2.5}$$

THEOREM 2.1. *In the special coordinates (2.2) obtained by combining the spacial comoving coordinates and membrane time the matrix of the pseudo-Riemannian metric (2.1) satisfies the relationships (2.5) and therefore becomes block-diagonal.*

Based on the signature $(+,-,-,-)$ of the pseudo-Riemannian metric in spacetime, we write the result of Theorem 2.1 as

$$G_{ij} = \left\Vert\begin{matrix} g_{00} & 0 & 0 & 0\\ 0 & -g_{11} & -g_{12} & -g_{13}\\ 0 & -g_{21} & -g_{22} & -g_{23}\\ 0 & -g_{31} & -g_{32} & -g_{33} \end{matrix}\right\Vert. \tag{2.6}$$

All components of the matrix (2.6) are functions of the coordinates (2.2). They can also be viewed as functions of the three spacial comoving coordinates and membrane time. No other coordinate systems will be considered in this book.

The quantities in the lower diagonal block of the matrix (2.6) define a three-dimensional Riemannian metric on the branes, which corresponds to the time-dependent Riemannian metric in the real physical universe:

$$g_{ij} = g_{ij}(t, x^1, x^2, x^3), \text{ where } 1 \leqslant i, j \leqslant 3. \tag{2.7}$$

Under the transformations of the comoving coordinates given by the formulas (3.3) in Chapter I they are transformed as follows:

$$\tilde{g}_{ij} = \sum_{k=1}^{3}\sum_{q=1}^{3} g_{kq}\,\frac{\partial x^k}{\partial \tilde{x}^i}\,\frac{\partial x^q}{\partial \tilde{x}^j}, \qquad g_{ij} = \sum_{k=1}^{3}\sum_{q=1}^{3} \tilde{g}_{kq}\,\frac{\partial \tilde{x}^k}{\partial x^i}\,\frac{\partial \tilde{x}^q}{\partial x^j}. \tag{2.8}$$

The formulas (2.8) are the transformation rules for a three-dimensional tensor field of valence $(0, 2)$.

The quantity g_{00} from the upper diagonal block of the matrix (2.6) is a scalar function on 3D-branes, which corresponds to a time-dependent scalar function in the real physical universe:

$$g_{00} = g_{00}(t, x^1, x^2, x^3). \tag{2.9}$$

Under the transformations of the comoving coordinates given by the formulas (3.3) in the first chapter the function (2.9) behaves like a scalar, i.e. it does not change. But under the membrane time scaling transformation given by the formulas (5.1) in the first chapter, it transforms as follows:

$$\tilde{g}_{00} = g_{00}\left(\frac{\partial t}{\partial \tilde{t}}\right)^2, \qquad g_{00} = \tilde{g}_{00}\left(\frac{\partial \tilde{t}}{\partial t}\right)^2. \tag{2.10}$$

The formulas (2.10) are transformation rules for a one-dimensional tensor field of valence $(0, 2)$.

The quantities (2.7) behave like scalars under the time scaling transformations given by the formulas (5.1) in the first chapter. They do not change. Only the time argument in them changes.

The scalar function (2.9) and the components of the metric (2.7) form a complete set of dynamic variables describing the gravitational field in the new theory. Due to the symmetry of the matrix (2.7) the number of such dynamic variables is 7. For comparison, in Einstein's theory of relativity the number of dynamic variables describing the gravitational field is 10.

§ 3. Einstein's equations.

We shall write Einstein's equations, which describe the gravitational field in Einstein's theory of relativity, as follows:

$$r_{ij} - \frac{r}{2}\,G_{ij} - \Lambda\,G_{ij} = \frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij}. \tag{3.1}$$

The form of Einstein's equations (3.1) is slightly different from that found in Wikipedia [45]. The main difference is the sign before Λ. This choice was made to match the notation with the book [3]. Due to the difference in sign that has arisen we choose here the value of the cosmological constant that differs in sign from that of Wikipedia [46]:

$$\Lambda \approx\ -1.0905\cdot 10^{-56}\ cm^{-2}. \tag{3.2}$$

In addition to (3.2) Einstein's equations (3.1) contain another constant γ. This is Newton's gravitational constant

$$\gamma \approx\ 6.674\cdot 10^{-8}\ cm^3\cdot g^{-1}\cdot s^{-2}, \tag{3.3}$$

which is a part of the law of universal gravitation (see [47] and [48]). The letter γ is used to denote the constant (3.3) to align the notation with the book [3].

The quantities T_{ij} in the right-hand side of the equation (3.1) are components of the energy-momentum tensor. They are determined by the fields of matter, including dark matter.

The r_{ij} quantities in (3.1) are the components of the Ricci tensor. They are calculated via the components of the metric (3.1) in several steps. First, the components of the metric connection are calculated — the Christoffel symbols:

$$\gamma^k_{ij} = \frac{1}{2} \sum_{s=0}^{3} G^{ks} \left(\frac{\partial G_{sj}}{\partial x^i} + \frac{\partial G_{is}}{\partial x^j} - \frac{\partial G_{ij}}{\partial x^s} \right), \tag{3.4}$$

see [49]. Then, using the Christoffel symbols (3.4), the components of the curvature tensor are calculated:

$$r^k_{isj} = \frac{\partial \gamma^k_{ji}}{\partial x^s} - \frac{\partial \gamma^k_{si}}{\partial x^j} + \sum_{q=0}^{3} \gamma^k_{sq}\,\gamma^q_{ji} - \sum_{q=0}^{3} \gamma^k_{jq}\,\gamma^q_{si}, \tag{3.5}$$

see [50]. The components of the Ricci tensor are obtained from the components of the curvature tensor (3.5) by contraction over the pair of indices k and s:

$$r_{ij} = \sum_{k=0}^{3} r^k_{ikj}, \tag{3.6}$$

see [51]. The scalar curvature is obtained by contracting the Ricci tensor (3.6) with the inverse metric tensor:

$$r = \sum_{i=0}^{3} \sum_{j=0}^{3} r_{ij}\, G^{ij}, \tag{3.7}$$

see [52]. The components of the inverse metric tensor in (3.4) and (3.7) are denoted by the letter G with upper indices. They form the matrix inverse to the matrix (2.1).

§ 4. Reduction of the four-dimensional Ricci tensor to 3D-branes.

The Ricci tensor (3.6) is included in the Einstein equations (3.1). Its reduction to 3D-branes consists in substituting the block-diagonal matrix (2.6) into the formulas (3.4), (3.5), and (3.6). The three-dimensional metric (2.7), which is a part of the matrix (2.6), defines its own set of Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{s=1}^{3} g^{ks}\left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right). \tag{4.1}$$

Some of the Christoffel symbols (3.4) coincide with the Christoffel symbols (4.1). Namely, it can be shown that

$$\gamma^k_{ij} = \Gamma^k_{ij} \quad \text{for} \quad 1 \leqslant i, j, k \leqslant 3. \tag{4.2}$$

The remaining components of the Christoffel symbols (3.4) are calculated as follows:

$$\gamma^0_{ij} = \frac{g_{00}^{-1}}{2}\,\frac{\partial g_{ij}}{\partial x^0} \quad \text{for} \quad 1 \leqslant i, j \leqslant 3, \tag{4.3}$$

$$\begin{aligned} \gamma^k_{0j} = \gamma^k_{j0} &= \frac{1}{2}\sum_{s=1}^{3} g^{ks}\,\frac{\partial g_{sj}}{\partial x^0} = \\ &= \sum_{s=1}^{3} g_{00}\,g^{ks}\,\gamma^0_{sj} \quad \text{for} \quad 1 \leqslant k, j \leqslant 3, \end{aligned} \tag{4.4}$$

$$\gamma^q_{00} = \frac{1}{2}\sum_{s=1}^{3} g^{qs}\,\frac{\partial g_{00}}{\partial x^s} \quad \text{for} \quad 1 \leqslant q \leqslant 3, \tag{4.5}$$

$$\gamma^0_{q0} = \gamma^0_{0q} = \frac{1}{2}\,g_{00}^{-1}\,\frac{\partial g_{00}}{\partial x^q} \quad \text{for} \quad 1 \leqslant q \leqslant 3, \tag{4.6}$$

$$\gamma^0_{00} = \frac{1}{2}\,g_{00}^{-1}\,\frac{\partial g_{00}}{\partial x^0}. \tag{4.7}$$

The formulas (4.3), (4.4), (4.5), (4.6), and (4.7) are derived from (3.4) using the formula (2.6).

Next, we define the following quantities:

$$b_{ij}=\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}. \tag{4.8}$$

In this case we assume that the special coordinates (2.2) are chosen in (4.8). The quantities (4.8) are the components of the symmetric tensor field **b**. Raising indices in (4.8), we produce the following quantities:

$$b^k_j=\sum^3_{s=1}g^{ks}\,b_{sj}, \qquad\qquad b^{ij}=\sum^3_{s=1}b^i_s\,g^{sj}. \tag{4.9}$$

Using the above quantities (4.8) and (4.9), the formulas (4.3) and (4.4) can be rewritten as follows:

$$\gamma^0_{ij}=g^{-1}_{00}\,b_{ij}\quad\text{for}\quad 1\leqslant i,j\leqslant 3, \tag{4.10}$$

$$\gamma^k_{0j}=\gamma^k_{j\,0}=b^k_j\quad\text{for}\quad 1\leqslant k,j\leqslant 3. \tag{4.11}$$

To calculate the components of the Ricci tensor using the formula (3.6), not all components of the curvature tensor (3.5) are needed, but only those for which $s=k$:

$$r^k_{ikj}=\frac{\partial\gamma^k_{ji}}{\partial x^k}-\frac{\partial\gamma^k_{ki}}{\partial x^j}+\sum^3_{q=0}\gamma^k_{kq}\,\gamma^q_{ji}-\sum^3_{q=0}\gamma^k_{jq}\,\gamma^q_{ki}. \tag{4.12}$$

Applying (4.2), (4.10), and (4.11) to (4.12), we obtain

$$\begin{gathered}r^k_{ikj}=R^k_{ikj}+g^{-1}_{00}\,b^k_k\,b_{ij}\,-\\-\,g^{-1}_{00}\,b^k_j\,b_{ki}\quad\text{for}\quad 1\leqslant i,j,k\leqslant 3.\end{gathered} \tag{4.13}$$

Here R^k_{ikj} are the components of the three-dimensional curvature tensor. They are given by a formula similar to (3.5):

$$R^k_{isj} = \frac{\partial \Gamma^k_{ji}}{\partial x^s} - \frac{\partial \Gamma^k_{si}}{\partial x^j} + \sum_{q=1}^{3} \Gamma^k_{sq}\,\Gamma^q_{ji} - \sum_{q=1}^{3} \Gamma^k_{jq}\,\Gamma^q_{si}. \tag{4.14}$$

The components of the three-dimensional connection in (4.14) are given by the formula (4.1). And the three-dimensional Ricci tensor is given by the formula

$$R_{ij} = \sum_{k=1}^{3} R^k_{ikj}, \tag{4.15}$$

which is analogous to (3.6).

Let's consider the case $k = 0$ and $1 \leqslant i, j \leqslant 3$ in (4.12). In this case we have the following relationship:

$$r^0_{i\,0j} = \frac{\partial \gamma^0_{ji}}{\partial x^0} - \frac{\partial \gamma^0_{0i}}{\partial x^j} + \sum_{q=0}^{3} \gamma^0_{0q}\,\gamma^q_{ji} - \sum_{q=0}^{3} \gamma^0_{jq}\,\gamma^q_{0i}. \tag{4.16}$$

By applying the formulas (4.10), (4.6), (4.2), (4.7), and (4.11) to (4.16), we reduce the formula (4.16) to the form

$$\begin{gathered} r^0_{i\,0j} = g_{00}^{-1}\,\frac{\partial b_{ij}}{\partial x^0} - \frac{1}{2}\,g_{00}^{-1}\,\nabla_{ij}\,g_{00} - \frac{1}{2}\,g_{00}^{-2}\,\frac{\partial g_{00}}{\partial x^0}\,b_{ij} + \\ + \frac{1}{4}\,g_{00}^{-2}\,\nabla_i\,g_{00}\,\nabla_j\,g_{00} - \sum_{q=1}^{3} g_{00}^{-1}\,b_{jq}\,b^q_i \quad \text{for} \quad 1 \leqslant i, j \leqslant 3. \end{gathered} \tag{4.17}$$

Applying (4.13) and (4.17) to (3.6), we derive a formula for a part of the components of the four-dimensional Ricci tensor:

$$\begin{gathered} r_{ij} = g_{00}^{-1}\,\frac{\partial b_{ij}}{\partial x^0} - \frac{1}{2}\,g_{00}^{-1}\,\nabla_{ij}\,g_{00} - \frac{1}{2}\,g_{00}^{-2}\,\frac{\partial g_{00}}{\partial x^0}\,b_{ij} + \\ + \frac{1}{4}\,g_{00}^{-2}\,\nabla_i\,g_{00}\,\nabla_j\,g_{00} + R_{ij} + g_{00}^{-1}\sum_{k=1}^{3} b^k_k\,b_{ij} - \\ - g_{00}^{-1}\sum_{k=1}^{3}(b_{ki}\,b^k_j + b_{kj}\,b^k_i) \quad \text{for} \quad 1 \leqslant i, j \leqslant 3. \end{gathered} \tag{4.18}$$

Here ∇ is the sign of the covariant derivative with respect to the three-dimensional metric connection with components (4.1).

The next step is the case $i = 0$ and $1 \leqslant j, k \leqslant 3$. in (4.12). In this case we have the relationship

$$\begin{aligned} r^k_{0kj} = \frac{\partial b^k_j}{\partial x^k} - \frac{\partial b^k_k}{\partial x^j} + \sum_{q=1}^{3} \Gamma^k_{kq}\, b^q_j - \sum_{q=1}^{3} \Gamma^k_{jq}\, b^q_k + \\ + \frac{1}{2}\, g_{00}^{-1}\, b^k_k\, \frac{\partial g_{00}}{\partial x^j} - \frac{1}{2}\, g_{00}^{-1}\, b^k_j\, \frac{\partial g_{00}}{\partial x^k}. \end{aligned} \tag{4.19}$$

We shall add two terms to the formula (4.19) and change the order of the terms in it:

$$\begin{aligned} r^k_{0kj} = \frac{\partial b^k_j}{\partial x^k} + \sum_{q=1}^{3} \Gamma^k_{kq}\, b^q_j - \sum_{q=1}^{3} \Gamma^q_{kj}\, b^k_q + \frac{1}{2}\, g_{00}^{-1}\, b^k_k\, \nabla_j\, g_{00} - \\ - \frac{\partial b^k_k}{\partial x^j} - \sum_{q=1}^{3} \Gamma^k_{jq}\, b^q_k + \sum_{q=1}^{3} \Gamma^q_{jk}\, b^k_q - \frac{1}{2}\, g_{00}^{-1}\, b^k_j\, \nabla_k\, g_{00}. \end{aligned}$$

Because $\Gamma^q_{kj} = \Gamma^q_{jk}$ the added terms are cancelled. But they allow us to replace the partial derivatives with covariant ones:

$$\begin{aligned} r^k_{0kj} = \nabla_k\, b^k_j - \nabla_j\, b^k_k + \frac{1}{2}\, g_{00}^{-1}\, b^k_k\, \nabla_j\, g_{00} - \\ - \frac{1}{2}\, g_{00}^{-1}\, b^k_j\, \nabla_k\, g_{00} \quad \text{for} \quad 1 \leqslant k, j \leqslant 3. \end{aligned} \tag{4.20}$$

Next we consider the case $i = k = 0$ and $1 \leqslant j \leqslant 3$ in (4.12). In this case we get the vanishing

$$r^0_{00j} = 0 \quad \text{for} \quad 1 \leqslant j \leqslant 3. \tag{4.21}$$

Applying (4.20) and (4.21) to (3.6), we obtain

$$r_{0j} = \sum_{k=1}^{3} \bigl(\nabla_k\, b^k_j - \nabla_j\, b^k_k\bigr) + \sum_{k=1}^{3} \frac{b^k_k\, \nabla_j\, g_{00} - b^k_j\, \nabla_k\, g_{00}}{2\, g_{00}}. \tag{4.22}$$

Due to the symmetry of $r_{ij} = r_{j\,i}$ from (4.22) we derive

$$r_{i\,0} = \sum_{k=1}^{3}\Bigl(\nabla_k\, b^k_i - \sum_{k=1}^{3}\nabla_i\, b^k_k\Bigr) + \sum_{k=1}^{3}\frac{b^k_k\,\nabla_i\, g_{00} - b^k_i\,\nabla_k\, g_{00}}{2\, g_{00}}. \tag{4.23}$$

The next step is to compute the component r_{00} of the four-dimensional Ricci tensor. We choose $i = 0$, $j = 0$, and $1 \leqslant k \leqslant 3$ in (4.12). As a result of this choice we get

$$\begin{aligned} r^k_{0k0} = \frac{1}{2}\sum_{s=1}^{3} g^{ks}\,\nabla_{ks}\, g_{00} - \frac{g_{00}^{-1}}{4}\sum_{s=1}^{3} g^{ks}\,\nabla_k\, g_{00}\,\cdot \\ \cdot\,\nabla_s\, g_{00} + \frac{1}{2}\, g_{00}^{-1}\,\frac{\partial g_{00}}{\partial x^0}\, b^k_k - \frac{\partial b^k_k}{\partial x^0} - \sum_{q=1}^{3} b^k_q\, b^q_k. \end{aligned} \tag{4.24}$$

And the last case is $i = 0$, $j = 0$, $k = 0$ in (4.12). It yields

$$r^0_{000} = 0. \tag{4.25}$$

Applying the relationships (4.24) and (4.25) to (3.6), we obtain

$$\begin{aligned} r_{00} = \frac{1}{2}\sum_{k=1}^{3}\sum_{s=1}^{3} g^{ks}\,\nabla_{ks}\, g_{00} - \frac{g_{00}^{-1}}{4}\sum_{k=1}^{3}\sum_{s=1}^{3} g^{ks}\,\nabla_k\, g_{00}\,\cdot \\ \cdot\,\nabla_s\, g_{00} + \frac{1}{2}\, g_{00}^{-1}\,\frac{\partial g_{00}}{\partial x^0}\sum_{k=1}^{3} b^k_k - \sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0} - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k. \end{aligned} \tag{4.26}$$

The formulas (4.18), (4.22), (4.23), and (4.26) perform the desired reduction of the four-dimensional Ricci tensor to 3D-branes. They express its components (3.6) in special coordinates (2.2) through the components of the three-dimensional Ricci tensor (4.15), through the scalar function g_{00}, and through the components of the tensor field **b** defined by means of the formula (4.8). The components of the three-dimensional Riemannian metric (2.7) are also present in these expressions.

§ 5. Reduction of the scalar curvature to 3D-branes.

The four-dimensional scalar curvature is given by the formula (3.7). Taking into account (2.6), this formula can be rewritten as

$$r=r_{00}\,g_{00}^{-1}-\sum_{i=1}^{3}\sum_{j=1}^{3}r_{ij}\,g^{ij}. \tag{5.1}$$

Applying (4.18) and (4.26) to (5.1) and taking into account the notation (4.8), we obtain the formula

$$\begin{gathered}
r=g_{00}^{-2}\,\frac{\partial g_{00}}{\partial x^0}\sum_{k=1}^{3}b^k_k+g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_{kq}\,g_{00}-\\
-\frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}-2\,g_{00}^{-1}\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}-\\
-R-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q.
\end{gathered}\tag{5.2}$$

The quantity R in (5.2) is the three-dimensional scalar curvature. It is defined by the following formula:

$$R=\sum_{i=1}^{3}\sum_{j=1}^{3}R_{ij}\,g^{ij}. \tag{5.3}$$

The formula (5.3) is an analogue of the formula (3.6) for the four-dimensional scalar curvature. And the formula (5.2) obtained above realizes the desired reduction of the four-dimensional scalar curvature to 3D-branes.

§ 6. Reduction of Einstein's equations to 3D-branes.

In the right-hand side of Einstein's equations (3.1) we see the components of the energy-momentum tensor. However, we do not have any formulas for these components since we do not consider any specific types of matter in the universe. Therefore to reduce the components of the energy-momentum tensor to 3D-branes it suffices to consider them as written in special coordinates (2.2).

Now we are ready to perform the reduction of Einstein's equations (3.1) to 3D-branes. As a result of such a reduction Einstein's equations are divided into three groups. The first group of equations is the most numerous. It contains six equations numbered with two indices $1 \leqslant i, j \leqslant 3$. The second group of equations contains three equations, numbered with the index $1 \leqslant i \leqslant 3$. The third group of equations contains only one equation. Let's write the first group of reduced equations:

$$\begin{aligned}
&\frac{g_{00}^{-2}}{2}\left(g_{ij}\sum_{k=1}^{3} b_k^k - b_{ij}\right)\frac{\partial g_{00}}{\partial x^0} + \frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(g^{kq}\,g_{ij} - \\
&\quad - \delta_i^k\,\delta_j^q\Bigr)\,\nabla_{kq}\,g_{00} - \frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(g^{kq}\,g_{ij} - \delta_i^k\,\delta_j^q\Bigr)\,\cdot \\
&\cdot \nabla_k\,g_{00}\,\nabla_q\,g_{00} + g_{00}^{-1}\left(\frac{\partial b_{ij}}{\partial x^0} - \sum_{k=1}^{3}\frac{\partial b_k^k}{\partial x^0}\,g_{ij} - \sum_{k=1}^{3}(b_{ki}\,\cdot \right. \\
&\cdot b_j^k + b_{kj}\,b_i^k) - \frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} b_q^k\,b_k^q - \frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} b_k^k\,b_q^q + \\
&\left. \quad + \sum_{k=1}^{3} b_k^k\,b_{ij}\right) + R_{ij} - \frac{R}{2}\,g_{ij} + \Lambda\,g_{ij} = \frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij}.
\end{aligned} \tag{6.1}$$

Then we write the second group of reduced Einstein equations:

$$\begin{aligned}
&\sum_{k=1}^{3}\nabla_k\,b_i^k - \sum_{k=1}^{3}\nabla_i\,b_k^k + \frac{1}{2}\,g_{00}^{-1}\sum_{k=1}^{3} b_k^k\,\nabla_i\,g_{00} - \\
&\quad - \frac{1}{2}\,g_{00}^{-1}\sum_{k=1}^{3} b_i^k\,\nabla_k\,g_{00} = \frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{i\,0}.
\end{aligned} \tag{6.2}$$

And finally, we write down the only equation from the third group of reduced Einstein equations:

$$\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}(b^k_k\,b^q_q - b^k_q\,b^q_k) + \frac{R}{2}\,g_{00} - \Lambda\,g_{00} = \frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{00}. \tag{6.3}$$

The equations (6.1) are derived using (4.18) and (5.2). The equations (6.3) are derived using (4.26) and (5.2). The equations (6.2) are derived using (4.23).

§ 7. Gravitational field equations in the new theory.

The number of different Einstein equations does not change as a result of their reduction to 3D-branes. There are 10 of them, six of them are in the equations (6.1), three are in the equations (6.2), and one equation is in (6.3). The number of dynamical variables describing the gravitational field in the new theory is seven. Therefore three equations are excluded from the new theory. These are the equations (6.2). The equations (6.1) and (6.3) remain and constitute the system of equations of the gravitational field in the new theory. The choice of these equations will be justified below in Chapter III.

§ 8. Schwarzschild black holes in the new theory.

Schwarzschild black holes are defined by the Schwarzschild metric. This metric is a solution of the Einstein equations (3.1) with zero right-hand side and with the choice $\Lambda = 0$ in them. In our theory we do not replace the Einstein equations (3.1) with others. We only transform them into special coordinates (2.2) associated with the foliation of 3D-branes and exclude some of them from the theory. Therefore all solutions of the Einstein equations (3.1) remain solutions of the gravitational field equations (6.1) and (6.3) in the new theory after transforming them into special coordinates (2.2).

The Schwarzschild metric is diagonal in the coordinates in which it is traditionally written. Its diagonal components are determined by the following formulas:

$$
\begin{aligned}
&g_{00} = 1 - \frac{r_{\text{gr}}}{\rho}, && g_{11} = \frac{-1}{1 - \dfrac{r_{\text{gr}}}{\rho}}\,,\\
&g_{22} = -\rho^2, && g_{33} = -\rho^2\,\sin^2(\theta).
\end{aligned}
\tag{8.1}
$$

The diagonality of the metric (8.1) is consistent with the block diagonality of the matrix (2.6). The constant r_{gr} in (8.1) is called the gravitational radius of the Schwarzschild black hole.

The variables ρ and θ can be considered as spherical comoving coordinates on branes supplemented by one more comoving coordinate ϕ. They can be supplemented by membrane time t. With this understanding of the variables present and absent in (8.1), 3D-branes will be given by the equations of the form $t = \text{const}$, while the coordinates

$$
x^0 = c_{\text{gr}}\,t, \qquad x^1 = \rho, \qquad x^2 = \theta, \qquad x^3 = \phi. \tag{8.2}
$$

will be analogous to the coordinates (2.2).

The Schwarzschild metric is stationary, its components (8.1) do not depend on the membrane time t in (8.2). Therefore from the formula (4.8) we derive

$$
b_{ij} = 0. \tag{8.3}
$$

By direct calculations it can be shown that the four-dimensional Ricci tensor for the Schwarzschild metric (8.1) is identically zero:

$$
r_{ij} = 0. \tag{8.4}
$$

The same is true for the four-dimensional scalar curvature:

$$
r = 0. \tag{8.5}
$$

The formula (8.4) is derived using the formulas (3.4), (3.5), and (3.6). Then the formula (8.5) is derived using (3.7). From (8.4) and (8.5) it follows that the Schwarzschild metric (8.1) is a solution of Einstein's equations (3.1) with zero right-hand side and with the choice $\Lambda = 0$ in them.

In the three-dimensional paradigm of the new theory, the Schwarzschild metric (8.1) is divided into a 3D metric

$$g_{11} = \frac{1}{1 - \dfrac{r_{\text{gr}}}{\rho}}, \qquad g_{22} = \rho^2, \qquad g_{33} = \rho^2 \sin^2(\theta) \tag{8.6}$$

and a separate scalar function

$$g_{00} = 1 - \frac{r_{\text{gr}}}{\rho}. \tag{8.7}$$

The metric (8.6) defines the components of the metric connection according to the formula (4.1):

$$\begin{aligned}
&\Gamma^1_{11} = \frac{r_{\text{gr}}}{2\,\rho\,(r_{\text{gr}} - \rho)}, && \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho},\\
&\Gamma^1_{22} = r_{\text{gr}} - \rho, && \Gamma^3_{23} = \cot\theta,\\
&\Gamma^1_{33} = (r_{\text{gr}} - \rho)\,\sin^2\theta, && \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{\rho},\\
&\Gamma^2_{33} = -\frac{\sin(2\,\theta)}{2}, && \Gamma^3_{32} = \cot\theta.
\end{aligned} \tag{8.8}$$

Using the connection components (8.8), we can calculate the components of the three-dimensional Ricci tensor for the metric (8.6) by means of the formulas (4.14) and (4.15). They form a diagonal 3×3 matrix with the elements

$$R_{11} = \frac{r_{\text{gr}}}{\rho^2\,(r_{\text{gr}} - \rho)}, \quad R_{22} = \frac{r_{\text{gr}}}{2\,\rho}, \quad R_{33} = \frac{r_{\text{gr}}\,\sin^2\theta}{2\,\rho} \tag{8.9}$$

in the diagonal. From (8.9), having calculated the scalar curvature by means of the formula (5.3), we find that it is zero:

$$R = 0. \tag{8.10}$$

In the equations (6.1) there is the gradient of the scalar function (8.7). Its components are easy to calculate:

$$\nabla_1\, g_{00} = \frac{r_{\text{gr}}}{\rho^2}\,, \qquad \nabla_2\, g_{00} = 0, \qquad \nabla_3\, g_{00} = 0. \tag{8.11}$$

In addition to the gradient (8.11) in the equations (6.1) there is the double gradient $\nabla_{ij}\, g_{00}$ of the scalar function (8.7):

$$\nabla_{ij}\, g_{00} = \frac{\partial^2 g_{00}}{\partial x^i\,\partial x^j} - \sum_{k=1}^{3} \Gamma^k_{ij}\,\frac{\partial g_{00}}{\partial x^k}. \tag{8.12}$$

The components of the double gradient (8.12) are also easy to calculate. They form a diagonal 3×3 matrix with the following entries in the diagonal:

$$\begin{aligned} \nabla_{11}\, g_{00} &= \frac{(4\,\rho - 3\,r_{\text{gr}})}{2\,(r_{\text{gr}} - \rho)\,\rho^3},\\ \nabla_{22}\, g_{00} &= \frac{r_{\text{gr}}\,(r_{\text{gr}} - \rho)}{\rho^2},\\ \nabla_{33}\, g_{00} &= \frac{r_{\text{gr}}\,(r_{\text{gr}} - \rho)\,\sin^2\theta}{\rho^2}. \end{aligned} \tag{8.13}$$

The terms with the gradient components (8.11) and with the double gradient components (8.13) in (6.1) have the form

$$\begin{aligned} A_{ij} &= \frac{g_{00}^{-2}}{4} \sum_{k=1}^{3}\sum_{q=1}^{3} \Bigl(g^{kq}\,g_{ij} - \delta^k_i\,\delta^q_j\Bigr)\,\nabla_k\, g_{00}\,\nabla_q\, g_{00},\\ B_{ij} &= \frac{g_{00}^{-1}}{2} \sum_{k=1}^{3}\sum_{q=1}^{3} \Bigl(g^{kq}\,g_{ij} - \delta^k_i\,\delta^q_j\Bigr)\,\nabla_{kq}\, g_{00}. \end{aligned} \tag{8.14}$$

The quantities (8.14) are the components of two 3×3 diagonal matrices with the diagonal elements

$$
\begin{aligned}
&A_{11} = 0, && B_{11} = \frac{r_{\text{gr}}}{(r_{\text{gr}} - \rho)\,\rho^2},\\
&A_{22} = \frac{r_{\text{gr}}\,(r_{\text{gr}} - \rho)}{\rho^2}, && B_{22} = \frac{r_{\text{gr}}\,(3\,r_{\text{gr}} - 2\,\rho)}{4\,(r_{\text{gr}} - \rho)\,\rho},\\
&A_{33} = \frac{r_{\text{gr}}\,(r_{\text{gr}} - \rho)\sin^2\theta}{\rho^2}, && B_{33} = \frac{r_{\text{gr}}\,(3\,r_{\text{gr}} - 2\,\rho)\sin^2\theta}{4\,(r_{\text{gr}} - \rho)\,\rho}.
\end{aligned}
\tag{8.15}
$$

Now we are ready to check the validity of equations (6.1), (6.2) and (6.3) for the metric (8.6) and the function (8.7). Due to (8.3) all components of the tensor field **b** in (6.1), (6.2), and (6.3) do vanish. It immediately follows that the equations (6.2) are satisfied provided that $T_{i\,0} = 0$. Further, through (8.10) and (8.3) we conclude that the equation (6.3) is satisfied provided $T_{00} = 0$ and due to the additional assumption $\Lambda = 0$. We proceed to equations (6.1). Due to the relationships (8.3), (8.10), and (8.14) obtained above, the equations (6.1) are reduced to the form

$$
B_{ij} - A_{ij} + R_{ij} + \Lambda\,G_{ij} = \frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij}.
\tag{8.16}
$$

Applying (8.15) and (8.9) to (8.16), we conclude that the equations (6.1) are satisfied under the condition $T_{ij} = 0$ and under the assumption that $\Lambda = 0$. The result obtained is formulated as the following theorem.

THEOREM 8.1. *The three-dimensional Schwarzschild metric* (8.6) *and the scalar function* (8.7) *satisfy the gravitational field equations* (6.1), (6.3), *and* (6.2) *with zero right-hand sides, i.e., in the absence of matter, within the framework of a cosmology with zero cosmological constant* $\Lambda = 0$.

§ 9. Coordinate covariance of the gravity equations.

Coordinate covariance of equations of geometric nature is usually defined as the preservation of the form of these equations when replacing some coordinates with others and simultaneously replacing the functions included in them with others according to certain rules. A typical example of coordinate covariant equations are differential equations for the components of tensor fields written using the operations of tensor multiplication, contraction, and covariant differentiation (see [53]). The gravitational field equations (6.1), (6.2), and (6.3) belong to this class of coordinate covariant equations. They exhibit the property of coordinate covariance with respect to replacing some comoving coordinates with others (see (3.3) in Chapter I).

§ 10. Covariance of the gravity equations with respect to scaling of membrane time.

The membrane time scaling transformations are given by the formulas (5.1) in the first chapter of the book. Taking into account (2.2), they can be written as follows:

$$\tilde{x}^0=\tilde{x}^0(x^0), \qquad\qquad x^0=x^0(\tilde{x}^0). \tag{10.1}$$

The transformations (10.1) do not affect the spacial comoving coordinates in (2.2). Therefore, we can write

$$\begin{cases}\tilde{x}^0=\tilde{x}^0(x^0),\\ \tilde{x}^1=x^1,\\ \tilde{x}^2=x^2,\\ \tilde{x}^3=x^3,\end{cases}\qquad\qquad \begin{cases}x^0=x^0(\tilde{x}^0),\\ x^1=\tilde{x}^1,\\ x^2=\tilde{x}^2,\\ x^3=\tilde{x}^3.\end{cases} \tag{10.2}$$

The four-dimensional metric (2.6) obeys the standard law of transformation for the components of a tensor field of valence $(0,2)$ under the transformations (10.2):

$$G_{ij}=\sum_{k=0}^{3}\sum_{q=0}^{3}\frac{\partial\tilde{x}^k}{\partial x^i}\,\frac{\partial\tilde{x}^q}{\partial x^j}\,\tilde{G}_{kq}, \tag{10.3}$$

The components of the energy-momentum tensor in the right-hand sides of the equations (6.1), (6.2), and (6.3) obey the same law. Therefore we can write a formula similar to (10.3):

$$T_{ij}=\sum_{k=0}^{3}\sum_{q=0}^{3}\frac{\partial\tilde{x}^k}{\partial x^i}\,\frac{\partial\tilde{x}^q}{\partial x^j}\,\tilde{T}_{kq}. \tag{10.4}$$

Due to the special form of the transformations (10.2) the formulas (10.3) preserve the block-diagonal form of the matrix (2.6). These formulas can be divided into spacial and temporal parts. The spacial part has the form

$$g_{ij}(x^0,x^1,x^2,x^3)=\tilde{g}_{ij}(\tilde{x}^0(x^0),x^1,x^2,x^3), \tag{10.5}$$

where $1\leqslant i,j\leqslant 3$. The temporal part has the form

$$g_{00}(x^0,x^1,x^2,x^3)=(\tilde{x}^0(x^0)')^2\,\tilde{g}_{00}(\tilde{x}^0(x^0),x^1,x^2,x^3). \tag{10.6}$$

Let's denote through ξ the derivative of the function $\tilde{x}^0(x^0)$ in (10.1). Then the formulas (10.5) and (10.6) can be rewritten as

$$g_{00}=\xi^2\,\tilde{g}_{00}, \qquad\qquad g_{ij}=\tilde{g}_{ij}. \tag{10.7}$$

Unlike (10.3), the formula (10.4) is divided not into two, but into three parts. Two of them have the form

$$T_{00}=\xi^2\,\tilde{T}_{00}, \qquad\qquad T_{ij}=\tilde{T}_{ij}\ \text{ for }\ 1\leqslant i,j\leqslant 3. \tag{10.8}$$

The third part of the formula (10.4) is written as follows:

$$T_{i0}=\xi\,\tilde{T}_{i0}\ \text{ and }\ T_{0i}=\xi\,\tilde{T}_{0i}\ \text{ for }\ 1\leqslant i\leqslant 3. \tag{10.9}$$

The transformations (10.7), (10.8) and (10.9) can be extended to all terms in the gravity equations (6.1), (6.2) and (6.3). From (10.7) we derive the equality

$$g^{ij}=\tilde{g}^{ij}. \tag{10.10}$$

Then, applying (10.7) and (10.10) to (4.8), we get

$$b_{ij} = \xi\,\tilde{b}_{ij}, \qquad\qquad b^k_q = \xi\,\tilde{b}^k_q. \tag{10.11}$$

Differentiating the first relationship (10.7) with respect to the variable x^0, we find that

$$\frac{\partial g_{00}}{\partial x^0} = \xi^3\,\frac{\partial \tilde{g}_{00}}{\partial \tilde{x}^0} + 2\,\xi\,\xi'\,\tilde{g}_{00}. \tag{10.12}$$

Similarly, differentiating the relationships (10.11) with respect to the variable x^0, we derive the relationships

$$\frac{\partial b_{ij}}{\partial x^0} = \xi^2\,\frac{\partial \tilde{b}_{ij}}{\partial \tilde{x}^0} + \xi'\,\tilde{b}_{ij}, \qquad \frac{\partial b^k_q}{\partial x^0} = \xi^2\,\frac{\partial \tilde{b}^k_q}{\partial \tilde{x}^0} + \xi'\,\tilde{b}^k_q. \tag{10.13}$$

The next step is to apply the second relationship (10.7), the relationships (10.2), and the relationship (10.10) to the formula (4.1). As a result, we obtain the transformation rule for the three-dimensional connection components Γ^k_{ij}:

$$\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij}. \tag{10.14}$$

The transformation rule (10.14) together with (10.2) yield

$$\nabla_i\,g_{00} = \xi^2\,\nabla_i\,\tilde{g}_{00}, \qquad\qquad \nabla_{ij}\,g_{00} = \xi^2\,\nabla_{ij}\,\tilde{g}_{00}. \tag{10.15}$$

Similarly, applying the relationships (10.14) and (10.2) to (10.11), we get the following formulas:

$$\nabla_i b_{kq} = \xi\,\nabla_i \tilde{b}_{ij}, \qquad\qquad \nabla_i b^k_q = \xi\,\nabla_i \tilde{b}^k_q. \tag{10.16}$$

The transformations (10.2), (10.7), (10.10), and (10.14) applied to the relationships (4.14), (4.15), and (5.3) yield

$$R_{ij} = \tilde{R}_{ij}, \qquad\qquad R = \tilde{R}. \tag{10.17}$$

THEOREM 10.1. *The gravity equations* (6.1), (6.2), *and* (6.3) *are covariant with respect to the transformations* (10.2), (10.7), (10.8), (10.9), (10.10), (10.11), (10.12), (10.13), (10.14), (10.15), (10.16), *and* (10.17), *which are induced by the membrane time scaling transformations*(10.1).

The proof of Theorem 10.1 consists in direct calculations using the formulas listed in the theorem.

CHAPTER III

LAGRANGIAN APPROACH TO DERIVING THE GRAVITATIONAL FIELD EQUATIONS.

§ 1. Action integral for the gravitational field.

To maintain continuity between Einstein's theory of relativity and the new theory considered in this book we have retained the concept of spacetime, although we have deprived it of its status as a four-dimensional physical continuum (see § 2 in Chapter I). The action of the gravitational field in general relativity is given by the four-dimensional integral

$$S_{\text{gr}}=-\frac{c_{\text{gr}}^3}{16\,\pi\,\gamma}\int(r+2\,\Lambda)\,\sqrt{-\det G}\;d^4x, \tag{1.1}$$

see [3]. We shall use the action (1.1), rewriting it in a three-dimensional form in terms of comoving coordinates and membrane time (see § 3 and § 5 in Chapter I), as it was done in [29]. Due to (2.6) in Chapter II we obtain

$$\sqrt{-\det G}=\sqrt{g_{00}}\;\sqrt{\det g}\,. \tag{1.2}$$

Substituting (1.2) into the formula (1.1) yields

$$S_{\text{gr}}=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\iint(r+2\,\Lambda)\,\sqrt{\det g}\,\sqrt{g_{00}}\;d^3x\,dt. \tag{1.3}$$

For the four-dimensional scalar curvature r the formula (5.2) was derived in Chapter II. Taking into account the formulas (2.2) from Chapter II, this formula can be rewritten as follows:

$$
\begin{gathered}
r=g_{00}^{-2}\,\frac{\dot g_{00}}{c_{\text{gr}}}\sum_{k=1}^{3}b^k_k+g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_{kq}\,g_{00}\,-\\
-\,\frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}-2\,g_{00}^{-1}\sum_{k=1}^{3}\frac{\dot b^k_k}{c_{\text{gr}}}\,-\\
-\,R-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q.
\end{gathered}
\tag{1.4}
$$

Traditionally the action integrals of physical theories contain the dynamical variables of these theories and their first derivatives with respect to time. In the formula (1.4) we see a term with $\dot b^k_k$. Applying the formulas (4.8) and (2.2) from Chapter II, we obtain the following formula:

$$
\dot b_{ij}=\frac{1}{2\,c_{\text{gr}}}\,\ddot g_{ij}.
\tag{1.5}
$$

Due to (1.5) the term with $\dot b^k_k$ contains the second derivatives of the dynamic variables with respect to time. Such a term must be excluded from the action integral for the gravitational field (1.3). This was done in [29].

§ 2. Reduction of the action integral.

Let's select the first and the fourth terms on the right side of formula (1.4). When we substitute them into the integral (1.3), we get the following integral over time:

$$
I=\int\limits_v^u\left(g_{00}^{-2}\,\frac{\dot g_{00}}{c_{\text{gr}}}\sum_{k=1}^{3}b^k_k-2\,g_{00}^{-1}\sum_{k=1}^{3}\frac{\dot b^k_k}{c_{\text{gr}}}\right)\sqrt{\det g}\,\sqrt{g_{00}}\,dt.
\tag{2.1}
$$

The integral (2.1) can be transformed to the form

$$
I=\int\limits_v^u\left(g_{00}^{-3/2}\,\frac{\dot g_{00}}{c_{\text{gr}}}\sum_{k=1}^{3}b^k_k-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{\dot b^k_k}{c_{\text{gr}}}\right)\sqrt{\det g}\,dt.
\tag{2.2}
$$

Further transformation of the integral (2.2) using integration by parts yields the following formulas:

$$\begin{gathered} I=\int\limits_{v}^{u}\frac{\partial}{\partial t}\biggl(-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\biggr)\sqrt{\det g}\;dt=-2\,g_{00}^{-1/2}\,\cdot\\ \cdot\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\,\sqrt{\det g}\,\biggr|_{v}^{u}+\int\limits_{v}^{u}2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\,\frac{\partial(\sqrt{\det g}\,)}{\partial t}\,dt. \end{gathered}\tag{2.3}$$

The integral term in (2.3) can be transformed using Jacobi's formula for differentiating a determinant (see [54]):

$$\frac{\partial(\sqrt{\det g}\,)}{\partial t}=\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\frac{\partial g_{kq}}{\partial t}\,\sqrt{\det g}\,.\tag{2.4}$$

Applying the formula (2.4) and the formulas (4.8) and (2.2) from Chapter II to the integral (2.3), we get

$$\begin{gathered} I=-2\,g_{00}^{-1/2}\sum_{k=1}^{3}\frac{b^k_k}{c_{\text{gr}}}\,\sqrt{\det g}\,\biggr|_{v}^{u}+\\ +\int\limits_{v}^{u}2\,g_{00}^{-1/2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,\sqrt{\det g}\;dt. \end{gathered}\tag{2.5}$$

The non-integral term in the formula (2.3) and the same term in the formula (2.5) can be omitted since such terms do not affect the differential equations derived from the action integrals. Upon omitting the non-integral term in (2.5) due to (2.5) the action integral (1.3) is transformed to the form

$$S_{\text{gr}}=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\iint(\rho+2\,\Lambda)\,\sqrt{\det g}\,\sqrt{g_{00}}\;d^3x\,dt,\tag{2.6}$$

where the scalar function ρ is given by the formula

$$\begin{aligned}\rho = g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\,g_{00} - \frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\cdot\\ \cdot\,\nabla_k\,g_{00}\,\nabla_q\,g_{00} - R - g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\,b^q_k\,+\\ +\,g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\,b^q_q.\end{aligned}\tag{2.7}$$

Unlike the original action integral (1.3), the action integral (2.6) contains only first-order time derivatives of the dynamic variables g_{ij} and g_{00} associated with the gravitational field.

§ 3. Lagrangian of the gravitational field and Lagrangian of matter.

It is known that the action integrals in field theories are time integrals of Lagrangians, and Lagrangians are integrals of the densities of Lagrangians over spatial variables. Therefore, we write (2.6) in the following form:

$$S_{\text{gr}} = \int L_{\text{gr}}\,dt, \qquad\qquad L_{\text{gr}} = \int \mathcal{L}_{\text{gr}}\,\sqrt{\det g}\,d^3x. \tag{3.1}$$

Matter has its own action integral and its own Lagrangian. We write them as follows:

$$S_{\text{mat}} = \int L_{\text{mat}}\,dt, \qquad L_{\text{mat}} = \int \mathcal{L}_{\text{mat}}\,\sqrt{\det g}\,d^3x. \tag{3.2}$$

The density of the Lagrangian in (3.1) is given by the formula

$$\mathcal{L}_{\text{gr}} = -\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\sqrt{g_{00}}\,(\rho + 2\,\Lambda), \tag{3.3}$$

where ρ is taken from (2.7). The square root of g_{00} is inherited from the four-dimensional action. Therefore here in the three-dimensional approach we do not include it in (3.1) and refer it to the Lagrangian density (3.3).

Due to the formula (2.7) the density of the Lagrangian (3.3) depends on g_{00} and g_{ij}, and on the time derivatives of these dynamic variables. The time derivatives of the metric components g_{ij} are replaced by b_{ij} by virtue of the formula (4.8) and the formula (2.2) from Chapter II. Therefore

$$L_{\text{gr}} = L_{\text{gr}}(g, \dot{g}, \mathbf{g}, \mathbf{b}). \tag{3.4}$$

Here g and $\dot{g}$ represent g_{00} and $\dot{g}_{00}$, while $\mathbf{g}$ and $\mathbf{b}$ represent g_{ij} and b_{ij}. The density of the Lagrangian of matter may depend on some additional dynamical variables describing the state of matter. We denote these dynamical variables by $Q_1, \ldots, Q_n$ and their time derivatives by $\dot{Q}_1, \ldots, \dot{Q}_n$:

$$\dot{Q}_i = \frac{\partial Q_i}{\partial t}. \tag{3.5}$$

The relationships (3.5) are similar to the relationships (4.8) from Chapter II. Based on them we write

$$L_{\text{mat}} = L_{\text{mat}}(g, \dot{g}, \mathbf{g}, \mathbf{b}, \mathbf{Q}, \dot{\mathbf{Q}}). \tag{3.6}$$

Each argument in the argument lists of L_{gr} and L_{mat} in (3.4) and (3.6) represents not only the corresponding group of dynamic variables, but also some finite number of their derivatives of various orders with respect to the spatial variables x^1, x^2, x^3.

The total action integral of the gravitational field and matter is the sum of the integrals (3.1) and (3.2):

$$S = \int L\,dt, \qquad\qquad L = \int \mathcal{L}\,\sqrt{\det g}\;d^3x, \tag{3.7}$$

where

$$\mathcal{L}=\mathcal{L}_{\text{gr}}+\mathcal{L}_{\text{mat}}.\tag{3.8}$$

The next step in the development of the theory is to apply the principle of least action[1] (see [55]) to the action integral S in the formulas (3.7). Applying this principle formally, we obtain three groups of differential equations. The first group of equations is

$$\begin{gathered}
-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\biggl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\biggr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\frac{1}{2}\,\biggl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\biggr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\\
+\biggl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\biggr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=0,\ \ \text{where}\ \ 1\leqslant i,j\leqslant 3.
\end{gathered}\tag{3.9}$$

This group of equations is associated with the dynamic variables g_{ij} and b_{ij}. The second group of equations is associated with the dynamic variables g_{00} and $\dot{g}_{00}$:

$$\begin{gathered}
-\frac{\partial}{\partial t}\biggl(\frac{\delta\mathcal{L}}{\delta\dot{g}_{00}}\biggr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-c_{\text{gr}}\,\biggl(\frac{\delta\mathcal{L}}{\delta\dot{g}_{00}}\biggr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^q_q+\\
+\biggl(\frac{\delta\mathcal{L}}{\delta g_{00}}\biggr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=0.
\end{gathered}\tag{3.10}$$

This group of equations consists of the single equation (3.10). The third group of equations is related to matter. It is associated with the dynamic variables $Q_1,\,\ldots\,,Q_n$ and $\dot{Q}_1,\,\ldots\,,\dot{Q}_n$ and describes the dynamics of these variables over time:

$$\begin{gathered}
-\frac{\partial}{\partial t}\biggl(\frac{\delta\mathcal{L}}{\delta\dot{Q}_i}\biggr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},\dot{Q}}}-c_{\text{gr}}\,\biggl(\frac{\delta\mathcal{L}}{\delta\dot{Q}_i}\biggr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\sum_{q=1}^{3}b^q_q+\\
+\biggl(\frac{\delta\mathcal{L}}{\delta Q_i}\biggr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{b},\dot{\mathbf{Q}}}}=0,\ \ \text{where}\ \ 1\leqslant i\leqslant 3.
\end{gathered}\tag{3.11}$$

[1] The principle of least action would be more correctly called the principle of stationary action since minimal action is never required in fact.

The equations (3.9) and (3.10) describe the evolution of the gravitational field, while the equations (3.11) describe the evolution of matter.

Below we shall not transform the equations (3.11) since in this book the variables $Q_1, \ldots, Q_n$ are not specified and there are no specific formulas for the density of the Lagrangian of matter $\mathcal{L}_{\text{mat}}$ in (3.2) and (3.8). As for equations (3.9) and (3.10), we shall transform them below. Let us denote

$$\begin{aligned}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}} &= -\,\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}} -\\ &\quad-\,\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}},\end{aligned}\tag{3.12}$$

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}} = -\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{kq}}\,g_{ki}\,g_{qj}.\tag{3.13}$$

In addition to the formula (3.12) and the formula (3.13) we consider the following formulas:

$$\begin{aligned}\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}} &= -\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta \dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} -\\ &\quad-\,c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta \dot{g}_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}\Bigr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}},\end{aligned}\tag{3.14}$$

$$\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}} = -\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}\,g_{00}^2.\tag{3.15}$$

Using (3.7) and applying (3.12) to (3.9), we derive

$$\begin{aligned}&-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}} - \frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3} b^q_q +\\ &\qquad+\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = -\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}.\end{aligned}\tag{3.16}$$

Similarly, using (3.7) and applying the formula (3.14) to the equation (3.10), we derive the following equation:

$$\begin{aligned} -\frac{\partial}{\partial t}\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta\dot{g}_{00}}\biggr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} - c_{\text{gr}}\,\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta\dot{g}_{00}}\biggr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3} b^q_q\,+\\ +\,\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\biggr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = -\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}. \end{aligned} \tag{3.17}$$

Now it only remains to derive explicit expressions for the left hand sides in the equations (3.16) and (3.17). To do this we use the formulas (3.1), (3.3) and (2.7).

§ 4. Equations for the three-dimensional metric.

In implicit form the differential equations that we need for the three-dimensional metric g_{ij} are written as the Euler-Lagrange equations (3.16). To make them explicit we need to calculate the partial variational derivatives in the left hand side of the equations (3.16). We introduce small variations to the dynamic variables b_{ij} using the following formula:

$$\hat{b}_{ij} = b_{ij}(t,x^1,x^2,x^3) + \varepsilon\, h_{ij}(t,x^1,x^2,x^3). \tag{4.1}$$

Here $\varepsilon \to 0$ is a small parameter, and $h_{ij}(t,x^1,x^2,x^3)$ are arbitrary smooth functions with compact support (see [56]). In this case, the partial variational derivatives of the Lagrangian density $\mathcal{L}_{\text{gr}}$ with respect to b_{ij} are defined by the formula

$$\begin{aligned} \hat{L}_{\text{gr}} = L_{\text{gr}} + \varepsilon\int\sum_{i=1}^{3}\sum_{j=1}^{3}\biggl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\biggr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\,\cdot\\ \cdot\, h_{ij}\,\sqrt{\det g}\; d^3x + \ldots\,, \end{aligned} \tag{4.2}$$

where L_{gr} is taken from (3.1) and $\hat{L}_{\text{gr}}$ is the result of substituting $\hat{b}_{ig}$ for b_{ij} in L_{gr}. The density of the Lagrangian $\mathcal{L}_{\text{gr}}$ in (4.2) is given by (3.3). It depends on b_{ij} only through the last two terms in the right hand side of the formula (2.7) for ρ. Similar terms are present in the formula (2.6) in [18]. Therefore we can apply the formula (6.3) from [18] upon slightly modifying it:

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{8\,\pi\,\gamma}\left(b^{ij}-\sum_{k=1}^{3}b_k^k\,g^{ij}\right).\tag{4.3}$$

Now, according to (3.16), we must differentiate the partial variational derivative (4.3) with respect to time t:

$$\begin{aligned}-\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial}{\partial t}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}&=\frac{c_{\text{gr}}^3\,g_{00}^{-3/2}}{16\,\pi\,\gamma}\left(\frac{b^{ij}}{2}-\sum_{k=1}^{3}b_k^k\,\frac{g^{ij}}{2}\right)\cdot\\ \cdot\,\dot{g}_{00}-\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{16\,\pi\,\gamma}&\left(\frac{1}{c_{\text{gr}}}\,\dot{b}^{ij}-\sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot{b}_k^k\,g^{ij}+\sum_{k=1}^{3}2\,b_k^k\,b^{ij}\right).\end{aligned}\tag{4.4}$$

In deriving the formula (4.4) we used the following formula for differentiating the inverse matrix:

$$\dot{g}^{ij}=-\sum_{k=1}^{3}\sum_{q=1}^{3}g^{ik}\,\dot{g}_{kq}\,g^{qj}.\tag{4.5}$$

Along with (4.5), when deriving (4.4), we used the formulas (4.8) and (2.2) from Chapter II to calculate $\dot{g}_{kq}$.

The second term in the left hand side of the formula (3.16) is transformed as follows:

$$\begin{aligned}-\frac{1}{2}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b_q^q&=\\ =-\frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}b_k^k\,b^{ij}-\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q\,g^{ij}\right)&.\end{aligned}\tag{4.6}$$

The third term in the left hand side of (3.16) contains the partial variational derivative of $\mathcal{L}_{\text{gr}}$ with respect to g_{ij}. To calculate this partial variational derivative we introduce a small variation of the metric:

$$\hat{g}_{ij} = g_{ij}(t, x^1, x^2, x^3) + \varepsilon\, h_{ij}(t, x^1, x^2, x^3). \tag{4.7}$$

Despite the relationships (4.8) and (2.2) from Chapter II, the variations (4.1) and (4.7) are assumed to be independent. Here again $\varepsilon \to 0$ is a small parameter and $h_{ij}(t, x^1, x^2, x^3)$ are arbitrary smooth functions with compact support. The partial variational derivative of the Lagrangian density $\mathcal{L}_{\text{gr}}$ with respect to g_{ij} is defined by the formula

$$\begin{gathered}\hat{L}_{\text{gr}} = L_{\text{gr}} + \varepsilon \int \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\cdot\\ \cdot\, h_{ij}\,\sqrt{\det g}\;d^3x + \ldots\,.\end{gathered} \tag{4.8}$$

The second integral L_{gr} in (3.1) after substituting (3.3) into it and after applying (2.7) is broken down into six integrals:

$$L_{\text{gr}} = L_1 + L_2 + L_3 + L_4 + L_5 + L_6. \tag{4.9}$$

The first of these integrals has the form

$$L_1 = -\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int \sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\, g_{00}^{-1/2}\,\nabla_{kq}\, g_{00}\,\sqrt{\det g}\;d^3x. \tag{4.10}$$

The second term in the right side of (4.9) is similar to (4.10):

$$L_2 = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int \sum_{\substack{k=1\\ q=1}}^{3} g^{kq}\,\frac{g_{00}^{-3/2}}{2}\,\nabla_k\, g_{00}\,\nabla_q\, g_{00}\,\sqrt{\det g}\;d^3x. \tag{4.11}$$

The third term in the right hand side of the formula (4.9) contains the scalar curvature R:

$$L_3=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int g_{00}^{1/2}\,R\,\sqrt{\det g}\;d^3x. \tag{4.12}$$

The fourth and fifth terms in the right hand side of the formula (4.9) are similar to each other:

$$L_4=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,b_q^k\,b_k^q\,\sqrt{\det g}\;d^3x, \tag{4.13}$$

$$L_5=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,b_k^k\,b_q^q\,\sqrt{\det g}\;d^3x. \tag{4.14}$$

The sixth term in the right hand side of the formula (4.9) contains the cosmological constant:

$$L_6=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int g_{00}^{1/2}\,2\,\Lambda\,\sqrt{\det g}\;d^3x. \tag{4.15}$$

In order to obtain an explicit expression for the partial variational derivative in (4.8) we need to substitute the expression (4.7) for g_{ij} in each of the integrals (4.10), (4.11), (4.12), (4.13), (4.14), (4.15) and then expand each of them in terms of the small parameter ε up to first order terms.

In the formula (4.10) there is the second order covariant derivative $\nabla_{kq}\,g_{00}$. It is calculated through the components Γ_{kq}^s of the metric connection for the metric (2.7) from Chapter II:

$$\nabla_{kq}\,g_{00}=\frac{g_{00}}{\partial x^k\,\partial x^q}-\sum_{s=1}^{3}\Gamma_{kq}^s\,\frac{g_{00}}{\partial x^s}. \tag{4.16}$$

The connection components are given by the Levi-Civita formula:

$$\Gamma_{kq}^s=\frac{1}{2}\sum_{r=1}^{3}g^{sr}\left(\frac{\partial g_{rq}}{\partial x^k}+\frac{\partial g_{kr}}{\partial x^q}-\frac{\partial g_{kq}}{\partial x^r}\right) \tag{4.17}$$

(see § 7 from Chapter III in [53]). Applying the formula (4.7) to the quantity g^{sr} in (4.17), we obtain

$$\hat{g}^{sr} = g^{sr} - \varepsilon \sum_{k=1}^{3} \sum_{q=1}^{3} g^{sk}\, h_{kq}\, g^{qr} + \ldots . \tag{4.18}$$

By ellipses in (4.2), (4.8) and (4.18) we denote terms of higher orders in the small parameter ε. The formula (4.18) is similar to the formula (4.5). Applying the formulas (4.7) and (4.18) to (4.17), we derive the formula

$$\hat{\Gamma}^{s}_{kq} = \Gamma^{s}_{kq} + \frac{\varepsilon}{2} \sum_{r=1}^{3} g^{sr}\, (\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}) + \ldots . \tag{4.19}$$

Now we apply the formula (4.19) to (4.16). This yields

$$\begin{gathered}\hat{\nabla}_{kq}\, g_{00} = \nabla_{kq}\, g_{00} - \frac{\varepsilon}{2} \sum_{r=1}^{3} \sum_{s=1}^{3} g^{sr} \bigl(\nabla_k\, h_{rq} + \\ + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\bigr)\, \nabla_s\, g_{00} + \ldots .\end{gathered} \tag{4.20}$$

In addition to the second covariant derivative (4.16) the integral L_1 in (4.10) contains g^{kq} and the square root $\sqrt{\det g}$. The expression g^{kq} is transformed using the formula (4.18). And for the square root $\sqrt{\det g}$ we write

$$\sqrt{\det \hat{g}} = \sqrt{\det g} + \frac{\varepsilon}{2} \sum_{r=1}^{3} \sum_{s=1}^{3} g^{rs}\, h_{rs} \sqrt{\det g} + \ldots . \tag{4.21}$$

Like the time derivative in the formula (2.4), the above formula (4.21) is derived using Jacobi's formula for differentiating determinants (see [54]).

Now we apply the formulas (4.18), (4.20) and (4.21) to the integral (4.10). As a result we obtain the following formula:

$$\hat{L}_1 = L_1 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3} g_{00}^{-1/2}\left(g^{ik}\,g^{qj} - \frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot$$
$$\cdot\,\nabla_{kq}\,g_{00}\,h_{ij}\sqrt{\det g}\;d^3x + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{k=1\\q=1}}^{3}\sum_{\substack{r=1\\s=1}}^{3} g_{00}^{-1/2}\,\frac{g^{kq}\,g^{sr}}{2}\,\cdot \tag{4.22}$$
$$\cdot\,\bigl(\nabla_k\,h_{rq} + \nabla_q\,h_{kr} - \nabla_r\,h_{kq}\bigr)\,\nabla_s\,g_{00}\,\sqrt{\det g}\;d^3x + \ldots\,.$$

The second integral in the formula (4.22) is transformed by means of integration by parts:

$$\hat{L}_1 = L_1 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3} g_{00}^{-1/2}\left(g^{ik}\,g^{qj} - \frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot$$
$$\cdot\,\nabla_{kq}\,g_{00}\,h_{ij}\sqrt{\det g}\;d^3x - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3}(2\,g^{ik}\,g^{qj} - \tag{4.23}$$
$$-\,g^{ij}\,g^{kq})\,\nabla_{kq}(g_{00}^{1/2})\,h_{ij}\,\sqrt{\det g}\;d^3x + \ldots\,.$$

Integration by parts in spaces with a Riemannian metric is based on the following formula:

$$\int\limits_{\Omega}\sum_{k=1}^{3}\nabla_k z^k\sqrt{\det g}\,d^3x = \int\limits_{\partial\Omega} g(\mathbf{z},\mathbf{n})\,dS. \tag{4.24}$$

This formula (4.24) is a three-dimensional version of formula (4.14) from Chapter IV in [3]. Note that the second covariant derivative $\nabla_{kq}(g_{00}^{1/2})$ can be written as

$$\nabla_{kq}(g_{00}^{1/2}) = \frac{1}{2}\,g_{00}^{-1/2}\,\nabla_{kq}g_{00} - \frac{1}{4}\,g_{00}^{-3/2}\,\nabla_k g_{00}\,\nabla_q g_{00}. \tag{4.25}$$

Applying the relationship (4.25) to (4.23), we obtain

$$\hat{L}_1 = L_1 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3}\frac{g_{00}^{-3/2}}{2}\left(g^{ik}\,g^{qj}-\frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot$$
$$\cdot\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,. \tag{4.26}$$

The second integral (4.11) is simpler than the first one, since the covariant derivatives $\nabla_k\,g_{00}$ and $\nabla_q\,g_{00}$ do not use the connection components (4.17). Applying the formulas (4.18) and (4.21) to this integral, we derive

$$\hat{L}_2 = L_2 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3}\frac{g_{00}^{-3/2}}{2}\left(g^{ik}\,g^{qj}-\frac{1}{2}\,g^{kq}\,g^{ij}\right)\cdot$$
$$\cdot\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots\,. \tag{4.27}$$

The third integral (4.12) is the most complicated. It contains the three-dimensional scalar curvature R. The scalar curvature R is calculated in several steps. First, the curvature tensor is calculated. The components of the curvature tensor are determined by the following formula:

$$R^k_{qij} = \frac{\partial\Gamma^k_{jq}}{\partial r^i} - \frac{\partial\Gamma^k_{iq}}{\partial r^j} + \sum_{s=1}^{3}\Gamma^k_{is}\,\Gamma^s_{jq} - \sum_{s=1}^{3}\Gamma^k_{js}\,\Gamma^s_{iq} \tag{4.28}$$

(see the formula (1.1) in chapter V of the book [3]. Next, the formula (4.19) is applied to (4.28). This yields

$$\hat{R}^k_{qij} = R^k_{qij} + \varepsilon\,\left(\nabla_i Y^k_{jq} - \nabla_j Y^k_{iq}\right) + \ldots\,, \tag{4.29}$$

where the following notations are made:

$$Y^s_{kq} = \frac{1}{2}\sum_{r=1}^{3} g^{sr}\,\left(\nabla_k\,h_{rq} + \nabla_q\,h_{kr} - \nabla_r\,h_{kq}\right). \tag{4.30}$$

The Ricci tensor is calculated using the curvature tensor. The components of the Ricci tensor are defined by the formula

$$R_{qj}=\sum_{k=1}^{3}R^{k}_{qkj}. \tag{4.31}$$

Applying the formula (4.29) to the formula (4.31), we obtain

$$\hat{R}_{qj}=R_{qj}+\varepsilon\sum_{k=1}^{3}\bigl(\nabla_k Y^k_{jq}-\nabla_j Y^k_{kq}\bigr)+\ldots\,. \tag{4.32}$$

The scalar curvature is calculated using the Ricci tensor

$$R=\sum_{q=1}^{3}\sum_{j=1}^{3}g^{qj}\,R_{qj}. \tag{4.33}$$

Applying (4.18) and (4.32) to the formula (4.33), we obtain

$$\hat{R}=R-\varepsilon\sum_{i=1}^{3}\sum_{j=1}^{3}R^{ij}\,h_{ij}+\varepsilon\sum_{k=1}^{3}\nabla_k Z^k+\ldots\,, \tag{4.34}$$

where the following notations are made:

$$Z^k=\sum_{q=1}^{3}\sum_{j=1}^{3}\bigl(g^{jq}\,Y^k_{jq}-g^{kq}\,Y^j_{jq}\bigr). \tag{4.35}$$

Now we are ready to apply the formula (4.34) to the third integral L_3 in (4.12). Along with the formula (4.34) we apply the formula (4.21). As a result, we obtain

$$\begin{gathered}\hat{L}_3=L_3-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}g_{00}^{1/2}\left(R^{ij}-\frac{R}{2}\,g^{ij}\right)h_{ij}\,\cdot\\ \cdot\sqrt{\det g}\;d^3x+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}g_{00}^{1/2}\,\nabla_k Z^k\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered} \tag{4.36}$$

The second integral in the right side of the formula (4.36) is transformed by means of integration by parts:

$$
\begin{aligned}
\hat{L}_3 = L_3 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3} g_{00}^{1/2}\left(R^{ij}-\frac{R}{2}\,g^{ij}\right)h_{ij}\,\cdot\\
\cdot\sqrt{\det g}\,d^3x - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3} Z^k\,\nabla_k(g_{00}^{1/2})\sqrt{\det g}\,d^3x+\ldots\,.
\end{aligned}
\tag{4.37}
$$

In order to make the second integral in (4.37) explicit, we evaluate Z^k explicitly by substituting (4.30) into (4.35). This yields

$$
Z^k = \sum_{q=1}^{3}\nabla_q\,h^{kq} - \sum_{q=1}^{3}\sum_{r=1}^{3} g^{kq}\,\nabla_q\,h^r_r.
\tag{4.38}
$$

Before substituting the formula (4.38) into (4.37) we transform this formula in the following way:

$$
Z^k = \sum_{q=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3}\bigl(g^{ki}\,g^{jq}\,\nabla_q\,h_{ij} - g^{kq}\,g^{ij}\,\nabla_q\,h_{ij}\bigr).
\tag{4.39}
$$

Now we substitute the formula (4.39) into the formulas (4.37) and apply integration by parts:

$$
\begin{aligned}
\hat{L}_3 = L_3 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3} g_{00}^{1/2}\left(R^{ij}-\frac{R}{2}\,g^{ij}\right)h_{ij}\,\cdot\\
\cdot\sqrt{\det g}\,d^3x + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(g^{ki}\,g^{jq}\,-\\
-\,g^{kq}\,g^{ij}\bigr)\,\nabla_{kq}(g_{00}^{1/2})\,h_{ij}\sqrt{\det g}\,d^3x+\ldots\,.
\end{aligned}
\tag{4.40}
$$

The integral L_4 in (4.13) is much simpler than the previous one. This is because it does not contain the spatial derivatives of the metric g_{ij}. Before applying (4.18) and (4.21) to it we write the integral in (4.13) as follows:

$$L_4=\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,g^{ki}\,b_{iq}\,\cdot\\ \cdot\,g^{qj}\,b_{jk}\,\sqrt{\det g}\;d^3x. \tag{4.41}$$

Then, applying (4.18) and (4.21) to (4.41), we get

$$\hat{L}_4=L_4+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3}g_{00}^{-1/2}\,b_q^k\,b_k^q\,\frac{g^{ij}}{2}\,h_{ij}\,\sqrt{\det g}\;d^3x\,-\\ -\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{k=1}^{3}g_{00}^{-1/2}(b^{ik}\,b_k^j+b^{jk}\,b_k^i)\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots. \tag{4.42}$$

The integral L_5 in (4.14) is transformed in a similar way. First we rewrite it as follows:

$$L_5=-\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3}g_{00}^{-1/2}\,g^{ik}\,b_{ik}\,\cdot\\ \cdot\,g^{qj}\,b_{qj}\,\sqrt{\det g}\;d^3x. \tag{4.43}$$

Next, applying (4.18) and (4.21) to (4.43), we obtain

$$\hat{L}_5=L_5+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{k=1}^{3}2\,g_{00}^{-1/2}\,b_k^k\,b^{ij}\,h_{ij}\,\sqrt{\det g}\;d^3x\,-\\ -\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{\substack{i=1\\j=1}}^{3}\sum_{\substack{k=1\\q=1}}^{3}g_{00}^{-1/2}\,b_k^k\,b_q^q\,\frac{g^{ij}}{2}\,h_{ij}\,\sqrt{\det g}\;d^3x+\ldots. \tag{4.44}$$

The integral L_6 in (4.15) is the simplest of the six integrals in (4.9). Applying the formula (4.21) to it, we obtain

$$\hat{L}_6 = L_6 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{i=1}^{3}\sum_{j=1}^{3} g_{00}^{1/2}\,2\,\Lambda\,\frac{g^{ij}}{2}\,\cdot \\ \cdot\,h_{ij}\,\sqrt{\det g}\;d^3x + \ldots\,. \tag{4.45}$$

Now we can put together the formulas (4.26), (4.27), (4.40), (4.42), (4.44), and (4.45) and derive a formula for the desired partial variational derivative

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\right)_{\substack{g,\dot{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = \frac{c_{\text{gr}}^4\,g_{00}^{-1/2}}{16\,\pi\,\gamma}\left(\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\,b^q_k\,\frac{g^{ij}}{2}\,- \right.\\ -\sum_{k=1}^{3}(b^{ik}\,b^j_k + b^{jk}\,b^i_k) + \sum_{k=1}^{3} 2\,b^k_k\,b^{ij} - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\,\cdot \\ \left.\cdot\,b^q_q\,\frac{g^{ij}}{2}\right) - \frac{c_{\text{gr}}^4\,g_{00}^{1/2}}{16\,\pi\,\gamma}\left(R^{ij} - \frac{R}{2}\,g^{ij} + \Lambda\,g^{ij}\right) + \\ + \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(g^{ki}\,g^{jq} - g^{kq}\,g^{ij}\bigr)\,\nabla_{kq}(g_{00}^{1/2}). \tag{4.46}$$

The next step is to put together the formulas (4.4), (4.6), and (4.46) and substitute them into the equation (3.15). This results in the following equation:

$$\frac{g_{00}^{-2}}{2\,c_{\text{gr}}}\left(\sum_{k=1}^{3} b^k_k\,g^{ij} - b^{ij}\right)\dot{g}_{00} + \frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\,g^{ij}\,- \\ - g^{ki}\,g^{jq})\,\nabla_{kq}\,g_{00} - \frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}\bigl(g^{kq}\,g^{ij} - g^{ki}\,g^{jq}\bigr)\,\cdot \\ \cdot\,\nabla_k\,g_{00}\,\nabla_q\,g_{00} + g_{00}^{-1}\left(\frac{1}{c_{\text{gr}}}\,\dot{b}^{ij} - \sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot{b}^k_k\,g^{ij} + \sum_{k=1}^{3}(b^{ik}\,\cdot\right. \tag{4.47}$$

$$\cdot\, b^j_k + b^{jk}\, b^i_k) - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k\, \frac{g^{ij}}{2} - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q\, \frac{g^{ij}}{2} +$$

$$+ \sum_{k=1}^{3} b^k_k\, b^{ij}\Biggr) + R^{ij} - \frac{R}{2}\, g^{ij} + \Lambda\, g^{ij} = \frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\, g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}.$$

The equation (4.47) and the equation (6.1) from Chapter II differ in the placement of the indices i and j. To compare the equation (4.47) with the equation (6.1) in Chapter II we need to lower the indices i and j in the equation (4.47). In performing this procedure we use the relationship

$$\dot{b}^{ij} = \sum_{k=1}^{3}\sum_{q=1}^{3} g^{ik}\, \dot{b}_{kq}\, g^{qj} - \sum_{k=1}^{3} 2\, c_{\text{gr}}\, (b^{ik}\, b^j_k + b^{jk}\, b^i_k). \tag{4.48}$$

In order to derive the relationship (4.48) the formulas (4.8) and (2.2) from Chapter II are used. Now, applying the formulas (4.48) and (3.13) to the equation (4.47) when lowering the indices i and j, we obtain the following equation:

$$\frac{g_{00}^{-2}}{2\,c_{\text{gr}}}\Biggl(\sum_{k=1}^{3} b^k_k\, g_{ij} - b_{ij}\Biggr)\,\dot{g}_{00} + \frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\, g_{ij} -$$

$$-\delta^k_i\,\delta^q_j)\,\nabla_{kq}\, g_{00} - \frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}(g^{kq}\, g_{ij} - \delta^k_i\,\delta^q_j)\,\cdot$$

$$\cdot\,\nabla_k\, g_{00}\,\nabla_q\, g_{00} + g_{00}^{-1}\Biggl(\frac{1}{c_{\text{gr}}}\,\dot{b}_{ij} - \sum_{k=1}^{3}\frac{1}{c_{\text{gr}}}\,\dot{b}^k_k\, g_{ij} - \sum_{k=1}^{3}(b_{ki}\,\cdot \tag{4.49}$$

$$\cdot\, b^k_j + b_{kj}\, b^k_i) - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k\,\frac{g_{ij}}{2} - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q\,\frac{g_{ij}}{2} +$$

$$+ \sum_{k=1}^{3} b^k_k\, b_{ij}\Biggr) + R_{ij} - \frac{R}{2}\, g_{ij} + \Lambda\, g_{ij} = -\frac{16\,\pi\,\gamma}{c_{\text{gr}}^4\, g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}}.$$

Comparing (4.49) with the equation (6.1) in Chapter II, we get

$$T_{ij} = -\frac{2}{g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}} \quad \text{for} \quad 1 \leqslant i, j \leqslant 3. \tag{4.50}$$

In the left hand side of the relationship (4.50) we see the components of the energy-momentum tensor that enter the equation (6.1) in Chapter II. The relationship (4.50) expresses these components of the energy-momentum tensor, which are a legacy of Einstein's four-dimensional theory, through a purely three-dimensional density of the Lagrangian of matter (3.6).

THEOREM 4.1. *The gravity equations* (6.1) *from Chapter II are equivalent to the Euler-Lagrange equations* (3.16), *which are explicitly written in the form of the equations* (4.47) *or in the form of the equations* (4.49).

§ 5. Equation for the scalar field g_{00}.

The time-dependent scalar field g_{00} arises as the diagonal component of the block-diagonal four-dimensional metric inherited from Einstein's theory, see formulas (2.9) and (2.6) in Chapter II. It is positive due to the signature $(+,-,-,-)$ of the metric (2.6) in Chapter II. The field g_{00} is described by the Euler-Lagrange equation (3.17). Our goal here is to write this equation in a more explicit form.

Using the formulas (3.3) and (2.7), we easily note that the Lagrangian density $\mathcal{L}_{\text{gr}}$ does not depend on the time derivative $\dot{g}_{00}$. Therefore we have the relationship

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta\dot{g}_{00}}\right)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = 0. \tag{5.1}$$

Due to (5.1) the Euler-Lagrange equation (3.17) reduces to

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\right)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = -\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}. \tag{5.2}$$

To calculate the partial variational derivative in the left hand side of (5.2) we consider a small variation of the scalar field g_{00}:

$$\hat{g}_{00} = g_{00}(t, x^1, x^2, x^3) + \varepsilon\, h(t, x^1, x^2, x^3). \tag{5.3}$$

Here $\varepsilon \to 0$ is a small parameter and $h(t, x^1, x^2, x^3)$ is an arbitrary smooth function with compact support (see [56]). The small variation (5.3) is applied to the Lagrangian L_{gr} in (3.1). After that the partial variational derivative of the Lagrangian density $\mathcal{L}_{\text{gr}}$ with respect to g_{00} is given by the formula

$$\hat{L}_{\text{gr}} = L_{\text{gr}} + \varepsilon \int \biggl(\frac{\delta \mathcal{L}_{\text{gr}}}{\delta g_{00}}\biggr)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} h \sqrt{\det g}\; d^3x + \ldots\,. \tag{5.4}$$

Like in § 4 here we split the Lagrangian L_{gr} into six parts using (4.9). The integrals L_1, L_2, L_3, L_4, L_5, and L_6 in (4.9) are given by (4.10), (4.11), (4.12), (4.13), (4.14), and (4.15). Applying (5.3) to the first of these six integrals, we obtain

$$\begin{aligned} \hat{L}_1 = L_1 &- \frac{c_{\text{gr}}^4\, \varepsilon}{16\,\pi\,\gamma} \int \sum_{\substack{k=1\\ q=1}}^{3} g^{kq}\, g_{00}^{-1/2}\, \nabla_{kq}\, h \sqrt{\det g}\; d^3x\, + \\ &+ \frac{c_{\text{gr}}^4\, \varepsilon}{16\,\pi\,\gamma} \int \sum_{\substack{k=1\\ q=1}}^{3} g^{kq}\, \frac{g_{00}^{-3/2}}{2}\, \nabla_{kq}\, g_{00}\, h \sqrt{\det g}\; d^3x + \ldots\,. \end{aligned} \tag{5.5}$$

The first integral in (5.5) is transformed integrating by parts:

$$\begin{aligned} \hat{L}_1 = L_1 - \frac{c_{\text{gr}}^4\, \varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3} \sum_{q=1}^{3} g^{kq}\, \frac{3\, g_{00}^{-5/2}}{4}\, \nabla_k\, g_{00}\, \nabla_q\, g_{00}\,\cdot \\ \cdot\, h \sqrt{\det g}\; d^3x + \frac{c_{\text{gr}}^4\, \varepsilon}{16\,\pi\,\gamma} \int \sum_{k=1}^{3} \sum_{q=1}^{3} g^{kq}\, g_{00}^{-3/2}\,\cdot \\ \cdot\, \nabla_{kq}\, g_{00}\, h \sqrt{\det g}\; d^3x + \ldots\,. \end{aligned} \tag{5.6}$$

Next we move on to the integral L_2 in (4.11) and apply the variation (5.3) to it. This gives

$$\begin{gathered}\hat{L}_2=L_2+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,g_{00}^{-3/2}\,\nabla_k\,g_{00}\,\nabla_q\,\cdot\\ \cdot\,h\,\sqrt{\det g}\;d^3x-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\frac{3\,g_{00}^{-5/2}}{4}\,\cdot\\ \cdot\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,h\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered}\tag{5.7}$$

The first integral in the formula (5.7) is transformed using integration by parts:

$$\begin{gathered}\hat{L}_2=L_2-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,g_{00}^{-3/2}\,\nabla_{kq}\,g_{00}\,\cdot\\ \cdot\,h\,\sqrt{\det g}\;d^3x+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}g^{kq}\,\frac{3\,g_{00}^{-5/2}}{4}\,\cdot\\ \cdot\,\nabla_k\,g_{00}\,\nabla_q\,g_{00}\,h\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered}\tag{5.8}$$

Next in line is the integral L_3 in (4.12). Applying the variation (5.3) to it, we obtain

$$\hat{L}_3=L_3+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\frac{g_{00}^{-1/2}}{2}\,R\,h\,\sqrt{\det g}\;d^3x+\ldots\,.\tag{5.9}$$

Next we pass to the integral L_4 in (4.13). Applying the variation (5.3) to this integral, we get

$$\begin{gathered}\hat{L}_4=L_4-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{g_{00}^{-3/2}}{2}\,b_q^k\,b_k^q\,\cdot\\ \cdot\,h\,\sqrt{\det g}\;d^3x+\ldots\,.\end{gathered}\tag{5.10}$$

The integral L_5 in (4.14) is treated in a similar way. Applying the variation (5.3) to it, we obtain

$$\begin{aligned}\hat{L}_5 = L_5 + \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\sum_{k=1}^{3}\sum_{q=1}^{3}\frac{g_{00}^{-3/2}}{2}\,b_k^k\,b_q^q\,\cdot\\ \cdot\,h\,\sqrt{\det g}\;d^3x+\ldots.\end{aligned} \tag{5.11}$$

And finally we arrive to the integral L_6 in (4.15). Applying the variation (5.3) to this integral, we get

$$\hat{L}_6 = L_6 - \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int g_{00}^{-1/2}\,\Lambda\,h\,\sqrt{\det g}\;d^3x. \tag{5.12}$$

Now we can put together the formulas (5.6), (5.8), (5.9), (5.10), (5.11), and (5.12) and apply them all to the formula (5.4). This gives us the following equality:

$$\begin{aligned}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\right)_{\substack{\dot{g},\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\frac{g_{00}^{-1/2}}{2}\,(R-2\Lambda)\,+\\ +\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\frac{g_{00}^{-3/2}}{2}\left(\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q-\sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q\right).\end{aligned} \tag{5.13}$$

Next, by substituting (5.13) into (5.2) we derive the equation

$$\begin{aligned}-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b_q^k\,b_k^q+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b_k^k\,b_q^q\,+\\ +\,\frac{R}{2}\,g_{00}-\Lambda\,g_{00} = -\frac{16\,\pi\,\gamma}{c_{\text{gr}}^4}\,g_{00}^{3/2}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{00}}.\end{aligned} \tag{5.14}$$

If we recall the relationship $g^{00} = g_{00}^{-1}$ and apply the formula (3.15) that follows from this relationship to the equation (5.14), then the equation (5.14) can be rewritten as follows:

$$
\begin{aligned}
-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\,b^q_k+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\,b^k_k\,b^q_q\,+\\
+\,\frac{R}{2}\,g_{00}-\Lambda\,g_{00}=\frac{16\,\pi\,\gamma}{c^4_{\text{gr}}\,g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}.
\end{aligned}
\tag{5.15}
$$

By comparing the equation (5.15) with the equation (6.3) in Chapter II, we derive the relationship

$$
T_{00}=\frac{2}{g_{00}^{1/2}}\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{00}}.
\tag{5.16}
$$

In the left hand side of the relationship (5.16) we see one of the components of the stress-energy tensor, which enters the equation (6.3) from Chapter II. The relationship (5.16) expresses this component T_{00} of the stress-energy tensor, which is a legacy of Einstein's four-dimensional theory, in terms of the purely three-dimensional density of the matter Lagrangian (3.6).

THEOREM 5.1. *The gravity equation* (6.3) *from Chapter II is equivalent to the Euler-Lagrange equation* (3.17), *which is explicitly written as the equation* (5.14) *or as the equations* (5.15).

Note that equations similar to those of (6.2) in Chapter II do not arise here within the Lagrangian approach. This justifies the choice made in Chapter II in favor of the equations (6.1) and (6.3) and the exclusion of the equations (6.2) from our new theory of gravity.

§ 6. Generalized coordinates and velocities.

Dynamic variables in the Lagrangian approach are often called generalized coordinates, while their time derivatives are called generalized velocities (see [57]). The dynamic variables for the gravitational field are the functions g_{ij} and g_{00}. In the case of the function g_{ij}, its time derivative is related to the function b_{ij}. From the formulas (4.8) and (2.2) in Chapter II, it follows that

$$b_{ij} = \frac{1}{2\,c_{\text{gr}}}\,\frac{\partial g_{ij}}{\partial t}. \tag{6.1}$$

Due to (6.1) the role of generalized velocities for the dynamic variables g_{ij} is played by the functions b_{ij}.

By analogy with the formula (6.1), the following notations were introduced in [31]:

$$b_{00} = \frac{\dot g_{00}}{c_{\text{gr}}} = \frac{1}{c_{\text{gr}}}\,\frac{\partial g_{00}}{\partial t}, \qquad\qquad b^0_0 = g^{-1}_{00}\,b_{00}. \tag{6.2}$$

Due to (6.2) the role of the generalized velocity for the dynamic variable g_{00} is played by the function b_{00}.

The dependence of the Lagrangian L_{gr} on the generalized coordinates and generalized velocities is conventionally represented by the formula (3.4). Taking into account (6.2), this formula now is rewritten in the following way:

$$L_{\text{gr}} = L_{\text{gr}}(g, b, \mathbf{g}, \mathbf{b}). \tag{6.3}$$

In order to describe matter above in § 3 the additional dynamic variables $Q_1, \ldots, Q_n$ and the corresponding generalized velocities (3.5) were introduced. We denote them through $W_1, \ldots, W_n$, i.e. we set

$$W_i = \dot Q_i = \frac{\partial Q_i}{\partial t}. \tag{6.4}$$

Taking into account (6.4), the formula (3.6) is rewritten as

$$L_{\text{mat}} = L_{\text{mat}}(g, b, \mathbf{g}, \mathbf{b}, \mathbf{Q}, \mathbf{W}). \tag{6.5}$$

The total Lagrangian is the sum of the gravitational field Lagrangian and the matter Lagrangian:

$$L = L_{\text{gr}} + L_{\text{mat}}. \tag{6.6}$$

The formula (6.6) follows from (3.7) and (3.8). Next, from the formulas (6.3), (6.5), and (6.6) we derive

$$L = L(g, b, \mathbf{g}, \mathbf{b}, \mathbf{Q}, \mathbf{W}). \tag{6.7}$$

Each argument in the argument lists of L_{gr}, L_{mat}, and L in (3.4), (6.5), and (6.7) represents not only the corresponding group of dynamic variables, but also a finite number of their derivatives of various orders with respect to the spacial variables x^1, x^2, x^3.

§ 7. Legendre transformation and the density of total energy.

The Legendre transformation is a change of dynamical variables in which generalized velocities are replaced by generalized momenta (see [58]). In the case of the present theory the generalized momenta are calculated as partial variational derivatives of the total Lagrangian (6.5):

$$\beta^{ij} = \left(\frac{\delta\mathcal{L}}{\delta b_{ij}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}, \quad \beta^{00} = \left(\frac{\delta\mathcal{L}}{\delta b_{00}}\right)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}, \quad P^{i} = \left(\frac{\delta\mathcal{L}}{\delta W_{i}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}. \tag{7.1}$$

Using generalized momenta (7.1), the total energy density is calculated by means of the formula

$$\mathcal{H} = \sum_{i=1}^{3}\sum_{j=1}^{3} \beta^{ij}\, b_{ij} + \beta^{00}\, b_{00} + \sum_{i=1}^{n} P^{i}\, W_{i} - \mathcal{L}. \tag{7.2}$$

Let Ω be a three-dimensional domain in the three-dimensional universe. The energy of the gravitational field and matter fields contained in this domain is given by the following integral:

$$E(\Omega) = \int\limits_{\Omega} \mathcal{H}\,\sqrt{\det g}\;d^3x. \tag{7.3}$$

Our main goal in the next section is to derive a formula for the time derivative of the integral (7.3).

§ 8. Energy conservation law.

Let us consider a small increment of the time variable t. We shall write it in the following form:

$$\hat{t} = t + \varepsilon. \tag{8.1}$$

Let's apply (8.1) to all dynamic variables:

$$\hat{g}_{ij} = g_{ij}(\hat{t}, x^1, x^2, x^3), \qquad \hat{Q}_i = Q_i(\hat{t}, x^1, x^2, x^3), \tag{8.2}$$

$$\hat{b}_{ij} = b_{ij}(\hat{t}, x^1, x^2, x^3), \qquad \hat{W}_i = W_i(\hat{t}, x^1, x^2, x^3), \tag{8.3}$$

$$\hat{\beta}^{ij} = \beta^{ij}(\hat{t}, x^1, x^2, x^3), \qquad \hat{P}^i = P^i(\hat{t}, x^1, x^2, x^3), \tag{8.4}$$

$$\hat{g}_{00} = g_{00}(\hat{t}, x^1, x^2, x^3), \qquad \hat{b}_{00} = b_{00}(\hat{t}, x^1, x^2, x^3), \tag{8.5}$$

$$\hat{\beta}^{00} = \beta^{00}(\hat{t}, x^1, x^2, x^3). \tag{8.6}$$

Applying the relationships (6.1) and (6.4) to (8.2), we obtain

$$\hat{g}_{ij} = g_{ij} + 2\,c_{\text{gr}}\,\varepsilon\,b_{ij} + \ldots\,, \qquad \hat{Q}_i = Q_i + \varepsilon\,W_i + \ldots\,. \tag{8.7}$$

In the case of (8.3) we use partial derivatives:

$$\hat{b}_{ij} = b_{ij} + \varepsilon\,\frac{\partial b_{ij}}{\partial t} + \ldots\,, \qquad \hat{W}_i = W_i + \varepsilon\,\frac{\partial W_i}{\partial t} + \ldots\,. \tag{8.8}$$

In the case of (8.4) and (8.6) we apply the relationships (7.1).

Applying them we derive the following formulas:

$$\begin{aligned}
&\hat{\beta}^{ij} = \beta^{ij} + \varepsilon\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g},\\ \mathbf{Q},\mathbf{W}}} + \ldots\,,\\
&\hat{P}^{i} = P^{i} + \varepsilon\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta W_{i}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}} + \ldots\,,\\
&\hat{\beta}^{00} = \beta^{00} + \varepsilon\,\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}} + \ldots\,.
\end{aligned}\tag{8.9}$$

In the case of the formula (8.5) we apply (6.2) and the partial time derivative of the function b_{00}:

$$\hat{g}_{00} = g_{00} + c_{\text{gr}}\,\varepsilon\,b_{00} + \ldots\,,\qquad \hat{b}_{00} = b_{00} + \varepsilon\,\frac{\partial b_{00}}{\partial t} + \ldots\,.\tag{8.10}$$

By ellipses in the formulas (8.7), (8.8), (8.9), (8.10) and further below we denote terms of higher order with respect to the small parameter $\varepsilon \to 0$.

In addition to the formulas (8.7), (8.8), (8.9), (8.10) we consider the following relationship:

$$\sqrt{\det\hat{g}} = \sqrt{\det g}\,\Bigl(1 + \varepsilon\,c_{\text{gr}}\sum_{k=1}^{3} b^{k}_{k} + \ldots\Bigr).\tag{8.11}$$

The relationship (8.11) is derived using the familiar Jacobi's formula for differentiating determinants (see [54]) and the relationship (6.1).

The next step is to apply the time variation (8.1) to the integral (7.3) taking into account (7.2):

$$\begin{aligned}
\hat{E}(\Omega) = \int\limits_{\Omega}\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij} + \beta^{00}\,b_{00} + \sum_{i=1}^{n} P^{i}\,W_{i}\Bigr)\,\cdot\\
\cdot\sqrt{\det\hat{g}}\;d^{3}x + \varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\,b_{ij} +
\end{aligned}$$

$$
\begin{aligned}
&+\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\,\frac{\partial b_{ij}}{\partial t}\Bigr)\,\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{n}\Bigl(\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\,W_i+\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\,\frac{\partial W_i}{\partial t}\Bigr)\,\cdot\\
&\cdot\sqrt{\det g}\;d^3x+\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\,b_{00}\,+ \qquad (8.12)\\
&+\Bigl(\frac{\delta\mathcal{L}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\,\frac{\partial b_{00}}{\partial t}\Bigr)\,\sqrt{\det g}\;d^3x-\hat{L}(\Omega)+\ldots\,.
\end{aligned}
$$

The last term $\hat{L}(\Omega)$ in (8.12) is determined by the Lagrangian density $\mathcal{L}$ in the formula (7.2):

$$
\hat{L}(\Omega)=\int\limits_{\Omega}\hat{\mathcal{L}}\sqrt{\det\hat{g}}\;d^3x. \tag{8.13}
$$

In order to transform the integral (8.13), it should be noted that the formulas (8.7), (8.8) and (8.10) are similar to small variations of the tensor fields $\mathbf{g}$ and $\mathbf{b}$, to small variations of the dynamic variables of matter $Q_1,\ldots,Q_n$ and $W_1,\ldots,W_n$, and also to small variations of the scalar fields g_{00} and b_{00} in the sense of the calculus of variations:

$$
\begin{aligned}
&\hat{g}_{ij}=g_{ij}+\varepsilon\,h_{ij}+\ldots\,, &&\hat{Q}_i=Q_i+\varepsilon\,h_i+\ldots\,,\\
&\hat{b}_{ij}=b_{ij}+\varepsilon\,\eta_{ij}+\ldots\,, &&\hat{W}_i=W_i+\varepsilon\,\eta_i+\ldots\,,\\
&\hat{g}_{00}=g_{00}+\varepsilon\,h_{00}+\ldots\,, &&\hat{b}_{00}=b_{00}+\varepsilon\,\eta_{00}+\ldots\,.
\end{aligned}
\tag{8.14}
$$

The functions h_{ij}, h_i, η_{ij}, η_i, h_{00}, and η_{00} in (8.14) are functions with compact support (see [56]). In the calculus of variations they apply to the integral over the entire universe:

$$
L=\int\mathcal{L}\,\sqrt{\det g}\;d^3x. \tag{8.15}
$$

Applying the small variations (8.14) to the integral (8.15) within the framework of the calculus of variations, we would write

$$\begin{gathered}\hat{L}=L+\varepsilon\int\Biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\delta\mathcal{L}}{\delta b_{ij}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\eta_{ij}\,+\\
+\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\delta\mathcal{L}}{\delta g_{ij}}\right)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}h_{ij}+\left(\frac{\delta\mathcal{L}}{\delta b_{00}}\right)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\eta_{00}\,+\\
+\left(\frac{\delta\mathcal{L}}{\delta g_{00}}\right)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}h_{00}+\sum_{i=1}^{n}\left(\frac{\delta\mathcal{L}}{\delta W_{i}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\eta_{i}\,+\\
+\sum_{i=1}^{n}\left(\frac{\delta\mathcal{L}}{\delta Q_{i}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{W}}}h_{i}\Biggr)\sqrt{\det g}\,d^{3}x+\ldots\,.\end{gathered}\tag{8.16}$$

The difference between the small variations in (8.7), (8.8), (8.10) and the small variations in (8.14) is that the small variations in (8.7), (8.8), (8.10) are not functions with compact support. For this reason, the analogue of formula (8.16) for the integral (8.13) will have an additional term containing an integral over the boundary of the domain Ω:

$$\begin{gathered}\hat{L}(\Omega)=L(\Omega)+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\delta\mathcal{L}}{\delta b_{ij}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\frac{\partial b_{ij}}{\partial t}\,\cdot\\
\cdot\sqrt{\det g}\,d^{3}x+\varepsilon\int\limits_{\Omega}\Biggl(\sum_{i=1}^{n}\left(\frac{\delta\mathcal{L}}{\delta W_{i}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\frac{\partial W_{i}}{\partial t}\,+\\
+\sum_{i=1}^{n}\left(\frac{\delta\mathcal{L}}{\delta Q_{i}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{W}}}W_{i}\Biggr)\sqrt{\det g}\,d^{3}x+\varepsilon\int\limits_{\Omega}\Biggl(\left(\frac{\delta\mathcal{L}}{\delta b_{00}}\right)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}\cdot\\
\cdot\frac{\partial b_{00}}{\partial t}+\left(\frac{\delta\mathcal{L}}{\delta g_{00}}\right)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}c_{\text{gr}}\,b_{00}\Biggr)\sqrt{\det g}\,d^{3}x\,+\\
+\,\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\delta\mathcal{L}}{\delta g_{ij}}\right)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}2\,c_{\text{gr}}\,b_{ij}\sqrt{\det g}\,d^{3}x\,+\end{gathered}\tag{8.17}$$

$$+\,\varepsilon\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3+\mathcal{J}^2\,dx^3\wedge dx^1+\\+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\sqrt{\det g}\,+\ldots\,.$$

The last term with the integral over the boundary of the domain in (8.17) is related to the energy flux across that boundary. We shall study this term in detail in the next section.

Now we return to the formula (8.12). The square root in the first integral of the formula (8.12) is transformed using the formula (8.11). We can then apply the formula (8.11) and the formula (8.17) to the formula (8.12). In doing so, we take into account the formulas (4.1) and the Euler-Lagrange equations (3.9), (3.10), and (3.11). In addition, we take into account the above notations (6.1), (6.2) and (6.4). As a result of the listed transformations, the formula (8.12) for the variation of the energy integral (7.3) is simplified and takes the following form:

$$\hat{E}(\Omega)=E(\Omega)-\varepsilon\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3+\\+\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\sqrt{\det g}\,+\ldots\,. \tag{8.18}$$

Now it is the moment to recall that the variation of the energy integral (8.18) arises as a result of the time increment (8.1). Therefore it can be calculated directly through the derivative of the integral (7.3) with respect to time:

$$\hat{E}(\Omega)=E(\Omega)+\varepsilon\,\frac{dE(\Omega)}{dt}+\ldots\,. \tag{8.19}$$

Comparing (8.18) and (8.19), we derive

$$\frac{d}{dt}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\,d^3x+\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3+\\+\,\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\sqrt{\det g}\,=0. \tag{8.20}$$

The surface integral of the second kind in (8.20) can be replaced by a surface integral of the first kind:

$$\frac{d}{dt}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}^1\,n_1+\mathcal{J}^2\,n_2+\mathcal{J}^3\,n_3\bigr)\,dS=0. \tag{8.21}$$

Here n_1, n_2, n_3 are the covariant components of the unit normal vector $\mathbf{n}$ perpendicular to the boundary of the domain $\partial\,\Omega$, and dS is the area element on this boundary. The quantities $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ in the formula (8.21) are interpreted as components of a vector field $\mathbf{J}$. This vector field itself is interpreted as the total energy flux density:

$$\frac{d}{dt}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\,\Omega}\sum_{i=1}^{3}\mathcal{J}^i\,n_i\,dS. \tag{8.22}$$

The equality (8.22) can be stated in the form of a theorem.

THEOREM 8.1. *The increment in the amount of the total energy of the gravitational field and matter fields contained in a three-dimensional domain Ω per unit time is exactly equal to the amount of energy entering the domain Ω per unit time through its boundary $\partial\,\Omega$.*

In order to transform the integral equality (8.22) into differential form, we apply the Ostrogradsky-Gauss formula (see [59]) together with the formula (8.11). This yields the following differential relationship:

$$\frac{\partial\mathcal{H}}{\partial t}+\sum_{q=1}^{3}c_{\text{gr}}\,\mathcal{H}\,b^q_q+\sum_{i=1}^{3}\nabla_{\!i}\mathcal{J}^i=0. \tag{8.23}$$

The first term in (8.23) is the time derivative of the total energy density of the gravitational field and matter fields. The third term is the divergence of the total energy flux density vector. These two terms are standard. The second term in (8.23) is the Hubble term. It arises because the volume of the domain Ω can change even if the boundaries of the domain are completely immobile due to the expansion or contraction of the three-dimensional space of the universe itself (see [60]).

§ 9. Flux density of the total energy.

The vector $\mathbf{J}$ with the components $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ arose in (8.17) when deriving an analogue of the formula (8.16) in which small variations of the dynamic variables are not functions with compact support. We know that the Lagrangian (6.7) depends not only on the functions listed in its arguments, but also on some finite number of their partial derivatives with respect to the spacial variables x^1, x^2, x^3. For this reason, we introduce the following notations for partial derivatives of the dynamic variables $Q_1, \ldots, Q_n$, and $W_1, \ldots, W_n$ describing matter and for partial derivatives of the dynamical variables g_{ij}, b_{ij}, g_{00}, and b_{00} describing the gravitational field:

$$Q_i[i_1\ldots i_s]=\frac{\partial Q_i}{\partial x^{i_1}\ldots\partial x^{i_s}},\ W_i[i_1\ldots i_s]=\frac{\partial W_i}{\partial x^{i_1}\ldots\partial x^{i_s}},\qquad (9.1)$$

$$g_{ij}[i_1\ldots i_s]=\frac{\partial g_{ij}}{\partial x^{i_1}\ldots\partial x^{i_s}},\ b_{ij}[i_1\ldots i_s]=\frac{\partial b_{ij}}{\partial x^{i_1}\ldots\partial x^{i_s}},\qquad (9.2)$$

$$g_{00}[i_1\ldots i_s]=\frac{\partial g_{00}}{\partial x^{i_1}\ldots\partial x^{i_s}},\ b_{00}[i_1\ldots i_s]=\frac{\partial b_{00}}{\partial x^{i_1}\ldots\partial x^{i_s}}.\qquad (9.3)$$

Let's select the quantity $b_{ij}[i_1\ldots i_s]$ from (9.2) and consider its occurrence in the Lagrangian (6.7). The variation of the quantity b_{ij} in (8.14) contributes to the variational expansion of the integral (8.13) in the form of the following expression:

$$I(\mathbf{b})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\ldots i_s]\,d^3x.\qquad (9.4)$$

Let ι_q denote a linear mapping acting upon differential 3-forms and generating differential 2-forms such that

$$\iota_q(dx^1\wedge dx^2\wedge dx^3)=\begin{cases} dx^2\wedge dx^3 & \text{if}\ \ q=1,\\ dx^3\wedge dx^1 & \text{if}\ \ q=2,\\ dx^1\wedge dx^2 & \text{if}\ \ q=3.\end{cases}\tag{9.5}$$

Then we can integrate (9.4) by parts. The result is written using the mapping introduced in (9.5):

$$\begin{gathered}\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_s]\,d^3x=\\ =\varepsilon\int\limits_{\partial\,\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_{s-1}]\,\cdot\\ \cdot\,\iota_{i_s}(dx^1\wedge dx^2\wedge dx^3)-\varepsilon\int\limits_{\Omega}\frac{\partial}{\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\cdot\\ \cdot\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_{s-1}]\,d^3x.\end{gathered}\tag{9.6}$$

The last term in (9.6) is similar to the first one. So we can continue integrating by parts in (9.6) iteratively. The result that we get upon several steps of integrating by parts is:

$$\begin{gathered}I(\mathbf{b})=\sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\cdot\\ \cdot\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\dots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\,\varepsilon\int\limits_{\Omega}(-1)^s\frac{\partial^s}{\partial x^{i_1}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}\,d^3x.\end{gathered}\tag{9.7}$$

The last term in (9.7) contributes to the volume integrals in (8.17). The preceding terms contribute to the boundary integral over $\partial\,\Omega$ in the end of the formula (8.17).

The variation of the metric g_{ij} from (9.2) contributes to the variational expansion of the integral in (8.13) via the expression

$$I(\mathbf{g})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{ij}[i_1\ldots i_s]\,d^3x. \tag{9.8}$$

Integrating by parts iteratively in the relationship (9.8), we derive a formula similar to that in (9.7):

$$\begin{aligned} I(\mathbf{g})=\sum_{r=1}^{s}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,h_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\\ +\varepsilon\int\limits_{\Omega}(-1)^s\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{ij}\,d^3x. \end{aligned} \tag{9.9}$$

The next two steps are similar to the previous two. The analogues of formulas (9.4) and (9.8) in the case of dynamic variables responsible for matter in (9.1) are as follows:

$$\begin{aligned} I(\mathbf{W})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_i[i_1\ldots i_s]\,d^3x,\\ I(\mathbf{Q})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_i[i_1\ldots i_s]\,d^3x. \end{aligned} \tag{9.10}$$

Integrating by parts in the relationships (9.10), we obtain formulas similar to those in (9.7) and (9.9):

$$I(\mathbf{W})=\sum_{r=1}^{s}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\cdot$$

$$\cdot\sqrt{\det g}\,\biggr)\,\eta_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+$$
$$+\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\biggr)\,\eta_i\,d^3x, \tag{9.11}$$

$$I(\mathbf{Q})=\sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\cdot$$
$$\cdot\sqrt{\det g}\,\biggr)\,h_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+ \tag{9.12}$$
$$+\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\biggr)\,h_i\,d^3x.$$

Next we move on to variations of g_{00} and b_{00} from (9.3). Here

$$I(b)=\varepsilon\int\limits_{\Omega}\biggl(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\biggr)\,\eta_{00}[i_1\ldots i_s]\,d^3x,$$
$$I(g)=\varepsilon\int\limits_{\Omega}\biggl(\frac{\partial\mathcal{L}}{\partial g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\biggr)\,h_{00}[i_1\ldots i_s]\,d^3x. \tag{9.13}$$

Integrating by parts iteratively the first of the integrals (9.13), we obtain the following formula:

$$I(b)=\sum^{s}_{r=1}\varepsilon\int\limits_{\partial\,\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\cdot$$
$$\cdot\sqrt{\det g}\,\biggr)\,\eta_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+ \tag{9.14}$$
$$+\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\biggr)\,\eta_{00}\,d^3x.$$

Similarly, integrating by parts iteratively the second of the integrals (9.13), we obtain a formula similar to (9.14):

$$
\begin{aligned}
I(g) = \sum_{r=1}^{s} \varepsilon \int\limits_{\partial\,\Omega} (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \ldots \partial x^{i_s}} \Bigl(\frac{\partial \mathcal{L}}{\partial g_{00}[i_1 \ldots i_s]} \cdot \\
\sqrt{\det g}\,\Bigr)\, h_{00}[i_1 \ldots i_{s-r}]\, \iota_{i_{s-r+1}} (dx^1 \wedge dx^2 \wedge dx^3) + \qquad (9.15)\\
+\,\varepsilon \int\limits_{\Omega} (-1)^s \frac{\partial^s}{\partial x^{i_1} \ldots \partial x^{i_s}} \Bigl(\frac{\partial \mathcal{L}}{\partial g_{00}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, h_{00}\, d^3x.
\end{aligned}
$$

The quantities η_{ij}, h_{ij}, η_i, h_i, η_{00}, h_{00} appearing in formulas (9.7), (9.9), (9.11), (9.12), (9.14), and (9.15) should be replaced by the following quantities:

$$
\eta_{ij} = \frac{\partial b_{ij}}{\partial t}, \qquad \eta_i = \frac{\partial W_i}{\partial t}, \qquad (9.16)
$$

$$
h_{ij} = 2\, c_{\text{gr}}\, b_{ij}, \qquad h_i = W_i, \qquad (9.17)
$$

$$
h_{00} = c_{\text{gr}}\, b_{00}, \qquad \eta_{00} = \frac{\partial b_{00}}{\partial t}. \qquad (9.18)
$$

The formulas (9.16), (9.17) and (9.18) are derived by comparing (8.14) with the formulas (8.7), (8.8) and (8.10).

The last step in calculating the components of the vector $\mathbf{J}$ is to collect the integrals over the boundary of the domain $\partial\,\Omega$ from all formulas (9.7), (9.9), (9.11), (9.12), (9.14), (9.15) in a single formula. Let N be the maximal order of partial derivatives of the form (9.1), (9.2), (9.3) contained in the Lagrangian $\mathcal{L}$. Then from the formulas (9.7), (9.9), (9.11), (9.12), (9.14), (9.15) and from the formula (8.17) we derive

$$
\begin{aligned}
\bigl(\mathcal{J}^1\, dx^2 \wedge dx^3 + \mathcal{J}^2\, dx^3 \wedge dx^1 + \mathcal{J}^3\, dx^1 \wedge dx^2\bigr) \sqrt{\det g} = \qquad (9.19)\\
= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{s=1}^{N} \sum_{r=1}^{s} (-1)^{r-1} \frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \ldots \partial x^{i_s}} \Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]} \times
\end{aligned}
$$

$$\times\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+$$
$$+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\,\times$$
$$\times\sqrt{\det g}\,\Bigr)\,h_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+$$
$$+\sum_{i=1}^{n}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\times$$
$$\times\sqrt{\det g}\,\Bigr)\,\eta_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+$$
$$+\sum_{i=1}^{n}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\times\tag{9.20}$$
$$\times\sqrt{\det g}\,\Bigr)\,h_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+$$
$$+\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial b_{00}[i_1\ldots i_s]}\,\times$$
$$\times\sqrt{\det g}\,\Bigr)\,\eta_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+$$
$$+\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{00}[i_1\ldots i_s]}\,\times$$
$$\times\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3).$$

Note that the substitutions (9.16), (9.17), and (9.18) should be applied to the formula (9.19) continued in (9.20) as well as to the previous formulas (9.4), (9.6), (9.7), (9.8), (9.9), (9.10), (9.11), (9.12), (9.13), (9.14), and (9.15). Note also that partial derivative operators of the form

$$\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}$$

are actually absent from those terms in (9.19) and (9.20) where $r = 1$. The same is true for all previous formulas where such operators are used.

§ 10. Density of the gravitational field energy.

Note that the total Lagrangian L of our theory in (3.7) is the sum of the gravitational field Lagrangian L_{gr} in (3.1) and the matter fields Lagrangian L_{mat} in (3.2):

$$L = L_{\text{gr}} + L_{\text{mat}}. \tag{10.1}$$

The same division into two terms holds for the density of the total Lagrangian $\mathcal{L}$, which is expressed by the formula (3.8) and from which the formula (10.1) follows. Therefore, applying (3.8) to (7.1), (7.2), and (7.3), we obtain

$$E(\Omega) = E_{\text{gr}}(\Omega) + E_{\text{mat}}(\Omega). \qquad \mathcal{H} = \mathcal{H}_{\text{gr}} + \mathcal{H}_{\text{mat}}. \tag{10.2}$$

Each of the terms $\mathcal{H}_{\text{gr}}$ and $\mathcal{H}_{\text{mat}}$ in (10.2) is given by its own formula, which follows from (7.2). In the case of the density of the gravitational field energy $\mathcal{H}_{\text{gr}}$ this formula has the form

$$\begin{aligned}\mathcal{H}_{\text{gr}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}} b_{ij} + \Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}} b_{00} \,+\\ +\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}} W_i - \mathcal{L}_{\text{gr}}.\end{aligned} \tag{10.3}$$

The density of the gravitational field Lagrangian $\mathcal{L}_{\text{gr}}$ in (10.3) is given by the formula (3.3), in which the parameter ρ is defined by the formula (2.7). The dynamic variables $Q_1, \ldots, Q_n$ and $W_1, \ldots, W_n$, which describe matter, are not included in the formulas (3.3) and (2.7). From this it follows that

$$\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta W_i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}} = 0. \tag{10.4}$$

In addition, we see that the quantity b_{00} is also not included in the formulas (3.3) and (2.7). Hence

$$\left(\frac{\delta\mathcal{L}}{\delta b_{00}}\right)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}=0. \tag{10.5}$$

After applying the formulas (10.4) and (10.5) to the formula (10.3) it simplifies and takes the form

$$\mathcal{H}_{\text{gr}}=\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}} b_{ij}-\mathcal{L}_{\text{gr}}. \tag{10.6}$$

We already have an explicit formula for the partial variational derivative from (10.6). This is the formula (4.3). Applying (4.3) to (10.6), we get the following formula:

$$\mathcal{H}_{\text{gr}}=\frac{c_{\text{gr}}^{4}\,g_{00}^{-1/2}}{8\,\pi\,\gamma}\left(\sum_{\substack{k=1\\ q=1}}^{3} b_{q}^{k}\,b_{k}^{q}-\sum_{\substack{k=1\\ q=1}}^{3} b_{k}^{k}\,b_{q}^{q}\right)-\mathcal{L}_{\text{gr}}. \tag{10.7}$$

The next step is to substitute (3.3) into (10.7) and to use the formula (2.7) for ρ. This gives

$$\begin{aligned}\mathcal{H}_{\text{gr}}=\frac{c_{\text{gr}}^{4}}{16\,\pi\,\gamma}\sqrt{g_{00}}\,\Biggl(g_{00}^{-1}\sum_{\substack{k=1\\ q=1}}^{3} b_{q}^{k}\,b_{k}^{q}-g_{00}^{-1}\sum_{\substack{k=1\\ q=1}}^{3} b_{k}^{k}\,b_{q}^{q}-{}\\ -R+2\,\Lambda+g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\,g_{00}-{}\\ -\frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{k}\,g_{00}\,\nabla_{q}\,g_{00}\Biggr).\end{aligned} \tag{10.8}$$

The resulting formula (10.8) is an explicit formula for the energy density of the gravitational field. The amount of energy of the gravitational field contained within a three-dimensional domain Ω is given by an integral similar to the integral (7.3):

$$E_{\text{gr}}(\Omega)=\int\limits_{\Omega}\mathcal{H}_{\text{gr}}\,\sqrt{\det g}\;d^3x. \tag{10.9}$$

The density of the gravitational field energy $\mathcal{H}_{\text{gr}}$ in the formula (10.9) is given by the explicit formula (10.8).

§ 11. Flux density of the gravitational field energy.

As we have already noted above, the total Lagrangian of our theory L is split into two parts — the Lagrangian of the gravitational field $\mathcal{L}_{\text{gr}}$ and the Lagrangian of the matter fields L_{mat}, see formula (10.1). The same holds for the Lagrangian density, see the formula (3.8). Applying (3.8) to the formula (9.19), which is continued in (9.20), we conclude that the flux density vector $\mathbf{J}$ is also split into two parts:

$$\mathbf{J}=\mathbf{J}_{\text{gr}}+\mathbf{J}_{\text{mat}}. \tag{11.1}$$

The first term in the formula (11.1) can be calculated explicitly. For this purpose we consider the integral (8.13), replacing the total Lagrangian density $\mathcal{L}$ with $\mathcal{L}_{\text{gr}}$ in it:

$$\hat{L}_{\text{gr}}(\Omega)=\int\limits_{\Omega}\hat{\mathcal{L}}_{\text{gr}}\,\sqrt{\det\hat{g}}\;d^3x. \tag{11.2}$$

From (11.2), using the formulas (8.7), (8.8), and (8.10), a formula similar to the formula (8.17) can be derived. In this case it is necessary to take into account the relationships (10.4) and (10.5). In addition to (10.4) and (10.5) there is another relationship

$$\left(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta Q_i}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{W}}}=0. \tag{11.3}$$

The relationship (11.3) follows from the fact that the formulas (3.3) and (2.7) do not contain $Q_1, \ldots, Q_n$. Taking this into account, we obtain the following relationship:

$$\begin{gathered}\hat{L}_{\text{gr}}(\Omega)=L_{\text{gr}}(\Omega)+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta b_{ij}}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{Q},\mathbf{W}}}\frac{\partial b_{ij}}{\partial t}\,\cdot\\ \cdot\sqrt{\det g}\;d^3x+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{ij}}\Bigr)_{\substack{g,b,\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}2\,c_{\text{gr}}\,b_{ij}\,\cdot\\ \cdot\sqrt{\det g}\;d^3x+\varepsilon\int\limits_{\Omega}\Bigl(\frac{\delta\mathcal{L}_{\text{gr}}}{\delta g_{00}}\Bigr)_{\substack{b,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\mathbf{W}}}c_{\text{gr}}\,b_{00}\,\cdot\\ \cdot\sqrt{\det g}\;d^3x+\varepsilon\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}^{1}_{\text{gr}}\,dx^2\wedge dx^3+\\ +\mathcal{J}^{2}_{\text{gr}}\,dx^3\wedge dx^1+\mathcal{J}^{3}_{\text{gr}}\,dx^1\wedge dx^2\bigr)\sqrt{\det g}\,+\ldots\,.\end{gathered}\tag{11.4}$$

From the derivation of the formula (9.19), which is continued in (9.20), we know that only terms with partial derivatives with respect to the spacial variables x^1, x^2, x^3 in the Lagrangian density lead to integrals over the boundary of the domain $\partial\,\Omega$. The term with $2\,\Lambda$ in (3.3) does not contain such derivatives. The terms with $b^k_q\,b^q_k$ and $b^k_k\,b^q_q$ in (2.7) also do not contain derivatives with respect to the spatial variables. There remain three terms:

1) the term with $\nabla_{kq}\,g_{00}$ in (2.7);
2) the term with $\nabla_k\,g_{00}\,\nabla_q\,g_{00}$ in (2.7);
3) the term with R in (2.7).

Let's start with the term with $\nabla_{kq}\,g_{00}$. This second covariant derivative itself is written using the components of the metric connection Γ^s_{kq} of the three-dimensional metric g_{ij}:

$$\nabla_{kq}\,g_{00}=\frac{\partial^2 g_{00}}{\partial x^k\,\partial x^q}-\sum_{s=1}^{3}\Gamma^s_{kq}\,\frac{\partial g_{00}}{\partial x^s}.\tag{11.5}$$

Applying the variation of the metric g_{ij} from (8.14) to Γ^i_{kq} in (8.3), we obtain the following expression:

$$\hat{\Gamma}^s_{kq} = \Gamma^s_{kq} + \frac{\varepsilon}{2}\sum_{r=1}^{3} g^{sr}\,\left(\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\right) + \ldots . \tag{11.6}$$

In addition to the variation of the metric we must take into account the variation of the scalar function g_{00} itself in (8.14). This leads to the formula

$$\begin{gathered}\hat{\nabla}_{kq}\,\hat{g}_{00} = \nabla_{kq}\, g_{00} + \nabla_{kq}\, h_{00} - \\ -\frac{\varepsilon}{2}\sum_{r=1}^{3}\sum_{s=1}^{3} g^{sr}\big(\nabla_k\, h_{rq} + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\big)\,\nabla_s\, g_{00} + \ldots .\end{gathered} \tag{11.7}$$

The formula (11.7) is derived using (11.6). Due to (11.7) the term with the second covariant derivative (11.5) contributes to the left hand side of the formula (11.4). Its contribution is expressed by the following two integrals:

$$L_1 = -\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega} g_{00}^{-1/2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\, h_{00}\,\sqrt{\det g}\; d^3x. \tag{11.8}$$

$$\begin{gathered}L_2 = \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\frac{g_{00}^{-1/2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g^{sr}\, g^{kq}\,\big(\nabla_k\, h_{rq} + \\ + \nabla_q\, h_{kr} - \nabla_r\, h_{kq}\big)\,\nabla_s\, g_{00}\,\sqrt{\det g}\; d^3x,\end{gathered} \tag{11.9}$$

Let's proceed to the term with $\nabla_k\, g_{00}\,\nabla_q\, g_{00}$ in (2.7). The covariant derivatives in this term do not use the connection components Γ^s_{kq}. Therefore the contribution of this term to the left hand side of (11.4) is expressed by the integral

$$L_3 = \frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega} g_{00}^{-3/2}\sum_{\substack{k=1\\ q=1}}^{3} g^{kq}\,\nabla_k\, g_{00}\,\nabla_q\, h_{00}\,\sqrt{\det g}\; d^3x. \tag{11.10}$$

The term with scalar curvature R in (2.7) is the most difficult to calculate. Applying the variation of the metric g_{ij} from (8.14) to R, we obtain the formula (4.34) which uses the notations (4.35) and (4.30). Due to (4.34) the contribution of the term with scalar curvature R in (2.7) to the left hand side of formula (8.2) is expressed by the integral

$$L_4=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega} g_{00}^{1/2}\sum_{k=1}^{3}\nabla_{k}Z^k\,\sqrt{\det g}\;d^3x. \tag{11.11}$$

Let's return to the integral (11.9). This integral can be simplified and can be written as the sum of two integrals:

$$\begin{aligned}L_2&=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}2\,\nabla_{k}\,h^{sk}\,\nabla_{s}\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x\,-\\&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}\sum_{r=1}^{3}g^{sr}\,\nabla_{r}\,h^{k}_{k}\,\nabla_{s}\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{11.12}$$

Applying integration by parts to the integrals (11.12), we obtain the following relationship:

$$\begin{aligned}L_2&=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}2\,h^{sk}\,\nabla_{s}\bigl(g_{00}^{1/2}\bigr)\,n_k\,dS\,-\\&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}2\,h^{sk}\,\nabla_{sk}\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x\,-\\&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}\sum_{r=1}^{3}g^{sr}\,\,h^{k}_{k}\,\nabla_{s}\bigl(g_{00}^{1/2}\bigr)\,n_r\,dS\,+\\&+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{s=1}^{3}\sum_{r=1}^{3}g^{sr}\,h^{k}_{k}\,\nabla_{sr}\bigl(g_{00}^{1/2}\bigr)\,\sqrt{\det g}\;d^3x.\end{aligned} \tag{11.13}$$

Here in (11.13) we denote by dS the area element on the boundary $\partial\,\Omega$ of the three-dimensional domain Ω, while n_k and n_r are the covariant components of the unit normal vector $\mathbf{n}$ to the boundary $\partial\,\Omega$ directed outward from the domain Ω.

Passing to the integral (11.8), we integrate it by parts twice. As a result, we obtain

$$
\begin{gathered}
L_1=-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega} g_{00}^{-1/2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{q}\,h_{00}\,n_k\,dS\,+\\
+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{k}\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,n_q\,dS\,-\\
-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\bigl(g_{00}^{-1/2}\bigr)\,h_{00}\,\sqrt{\det g}\;d^3x.
\end{gathered}
\tag{11.14}
$$

In (11.14), as in (11.13), dS denotes the area element on the boundary $\partial\,\Omega$, while n_k and n_q are the covariant components of the unit normal vector $\mathbf{n}$ to the boundary $\partial\,\Omega$.

Now we proceed to the integral (11.10). Note that this integral can be written as follows:

$$
\begin{gathered}
L_3=-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{k}\bigl(\,g_{00}^{-1/2}\bigr)\,\cdot\\
\cdot\,\nabla_{q}\,h_{00}\,\sqrt{\det g}\;d^3x.
\end{gathered}
\tag{11.15}
$$

Integrating by parts in (11.15), we derive the formula

$$
\begin{gathered}
L_3=-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{k}\bigl(\,g_{00}^{-1/2}\bigr)\,h_{00}\,n_q\,dS\,+\\
+\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\bigl(\,g_{00}^{-1/2}\bigr)\,h_{00}\,\sqrt{\det g}\;d^3x.
\end{gathered}
\tag{11.16}
$$

The next step is to transform the integral (11.11). Integrating it by parts, we get the following formula:

$$
\begin{gathered}
L_4=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega} g_{00}^{1/2}\sum_{k=1}^{3} Z^k\,n_k\,dS\,-\\
-\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\nabla_{\!k}\bigl(g_{00}^{1/2}\bigr)\,Z^k\,\sqrt{\det g}\,d^3x.
\end{gathered}
\tag{11.17}
$$

Earlier from the formulas (4.35) and (4.30) we obtained the expression (4.38) for Z^k. Here we apply the formula (4.38) to the second integral in (11.17). This yields

$$
\begin{gathered}
L_4=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega} g_{00}^{1/2}\sum_{k=1}^{3} Z^k\,n_k\,dS\,-\\
-\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\nabla_{\!k}\bigl(g_{00}^{1/2}\bigr)\,\nabla_{\!q}\,h^{kq}\sqrt{\det g}\,d^3x\,+\\
+\,\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_{\!k}\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,\nabla_{\!q}\,h^r_r\,\sqrt{\det g}\,d^3x.
\end{gathered}
$$

Next we apply integration by parts to the second and third integrals in the resulting formula. This gives

$$
\begin{gathered}
L_4=\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\biggl(\,\int\limits_{\partial\,\Omega} g_{00}^{1/2}\sum_{k=1}^{3} Z^k\,n_k\,dS-\int\limits_{\partial\,\Omega}\sum_{\substack{k=1\\q=1}}^{3}\nabla_{\!k}\bigl(g_{00}^{1/2}\bigr)\,\cdot\\
\cdot\,h^{kq}\,n_q\,dS+\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\nabla_{\!kq}\bigl(g_{00}^{1/2}\bigr)\,h^{kq}\sqrt{\det g}\,d^3x\biggr)\,+
\end{gathered}
$$

$$
\begin{aligned}
&+\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_k\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,h_r^r\,n_q\,dS\,-\\
&-\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\Omega}\sum_{k=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\nabla_{kq}\bigl(g_{00}^{1/2}\bigr)\,g^{kq}\,h_r^r\,\sqrt{\det g}\;d^3x.
\end{aligned}
\tag{11.18}
$$

As in the previous formulas, by dS in (11.18) we denote the area element on the boundary $\partial\,\Omega$ of the three-dimensional domain Ω, while n_k and n_q are the covariant components of the unit normal vector $\mathbf{n}$ to the boundary of the domain $\partial\,\Omega$ directed outward from the domain Ω.

Now let's recall that the surface integral of the second kind in (11.4) can be transformed into an integral of the first kind:

$$
\begin{aligned}
&\int\limits_{\partial\,\Omega}\bigl(\mathcal{J}_{\text{gr}}^1\,dx^2\wedge dx^3+\mathcal{J}_{\text{gr}}^2\,dx^3\wedge dx^1\,+\\
&+\mathcal{J}_{\text{gr}}^3\,dx^1\wedge dx^2\bigr)\sqrt{\det g}\;=\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\mathcal{J}^k\,n_k\,dS.
\end{aligned}
\tag{11.19}
$$

Due to (11.19) we can put together all of the integrals over the boundary of the domain $\partial\,\Omega$ from (11.13), (11.14), (11.16), and (11.18) and compare their sum with the right hand side of (11.19). This gives the following formula:

$$
\begin{aligned}
&\frac{c_{\text{gr}}^4\,\varepsilon}{16\,\pi\,\gamma}\int\limits_{\partial\,\Omega}\sum_{k=1}^{3}\biggl(\sum_{s=1}^{3}2\,h^{sk}\,\nabla_s\bigl(g_{00}^{1/2}\bigr)-\sum_{q=1}^{3}\sum_{s=1}^{3}g^{sk}\,\cdot\\
&\cdot\,h_q^q\,\nabla_s\bigl(g_{00}^{1/2}\bigr)-\sum_{q=1}^{3}g_{00}^{-1/2}\,g^{kq}\,\nabla_q\,h_{00}+\sum_{q=1}^{3}g^{kq}\,\cdot\\
&\cdot\,\nabla_q\bigl(g_{00}^{-1/2}\bigr)\,h_{00}-\sum_{q=1}^{3}g^{kq}\,\nabla_q\bigl(\,g_{00}^{-1/2}\bigr)\,h_{00}+g_{00}^{1/2}\,\cdot
\end{aligned}
\tag{11.20}
$$

$$\cdot Z^k - \sum_{q=1}^{3} \nabla_q\bigl(g_{00}^{1/2}\bigr)\, h^{kq} + \sum_{q=1}^{3}\sum_{r=1}^{3} \nabla_q\bigl(g_{00}^{1/2}\bigr)\, g^{kq}\,\cdot$$

$$\cdot\, h^r_r\biggr) = \varepsilon \int\limits_{\partial\,\Omega} \sum_{k=1}^{3} \mathcal{J}^k_{\text{gr}}\, n_k\, dS.$$

Applying the formula (4.38), from the formula (11.20) we determine the components of the vector $\mathbf{J}_{\text{gr}}$:

$$\begin{aligned}\mathcal{J}^k_{\text{gr}} = \frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\Biggl(\sum_{i=1}^{3} g_{00}^{1/2}\,\nabla_i\, h^{ik} - \sum_{i=1}^{3}\sum_{q=1}^{3} g_{00}^{1/2}\, g^{ik}\,\nabla_i\, h^q_q\, + \\ + \sum_{i=1}^{3} h^{ik}\,\nabla_i\bigl(g_{00}^{1/2}\bigr) - \sum_{i=1}^{3} g_{00}^{-1/2}\, g^{ik}\,\nabla_i\, h_{00}\Biggr).\end{aligned} \tag{11.21}$$

The final formula for $\mathcal{J}^k_{\text{gr}}$ is obtained from (11.21) after applying the formulas (9.17) and (9.18). It has the form

$$\begin{aligned}\mathcal{J}^k_{\text{gr}} = \frac{c_{\text{gr}}^5}{16\,\pi\,\gamma}\Biggl(\sum_{i=1}^{3} 2\, g_{00}^{1/2}\,\nabla_i\, b^{ik} - \sum_{i=1}^{3}\sum_{q=1}^{3} 2\, g_{00}^{1/2}\, g^{ik}\,\cdot \\ \cdot\,\nabla_i\, b^q_q + \sum_{i=1}^{3} 2\, b^{ik}\,\nabla_i\bigl(g_{00}^{1/2}\bigr) - \sum_{i=1}^{3} g_{00}^{-1/2}\, g^{ik}\,\nabla_i\, b_{00}\Biggr).\end{aligned} \tag{11.22}$$

In general, the energy flux of the gravitational field through a surface S is given by the formula

$$E(S) = \int\limits_{S} \sum_{k=1}^{3} \mathcal{J}^k_{\text{gr}}\, n_k\, dS. \tag{11.23}$$

The components $\mathcal{J}^k_{\text{gr}}$ of the vector $\mathbf{J}_{\text{gr}}$ in (11.23) are determined by the formula (11.22).

Theorem 8.1 states the total energy conservation law including the gravitational field energy and the energy of matter fields. There is no separate conservation law for the gravitational field energy. But in the formula for the total energy density (7.2) we can separate the part (10.8) responsible for the gravitational field. In the same way, from the vector of the total energy flux $\mathbf{J}$ defined by the formula (9.19), which is continued in (9.20), we can separate the part $\mathbf{J}_{\text{gr}}$ defined by the formula (11.22), which is responsible for the energy flux of the gravitational field.

L. D. Faddeev in [61] points out some problems with the definition of the energy for the gravitational field in Einstein's theory of relativity. In our new theory, as we can see in the formulas (10.8) and (11.22), there are no problems with the energy of the gravitational field.

CHAPTER IV

POINT PARTICLES IN A GRAVITATIONAL FIELD.

§ 1. Action integral for point particles.

Point particles or point masses in mechanics are usually considered to be particles whose size is negligibly small compared to the distances of their movements and whose internal structure and spatial orientation do not affect their motion. The location of a point particle in space is characterized by three coordinates x^1, x^2, x^3. The motion of a point particle is characterized by the fact that its coordinates x^1, x^2, x^3 depend on time:

$$x^1(t), \qquad x^2(t), \qquad x^3(t). \tag{1.1}$$

In our theory there are distinguished coordinate systems which are called comoving coordinates, see § 3 in Chapter I. In what follows by x^1, x^2, x^3 in (1.1) and everywhere below we mean coordinates in one of such comoving coordinate systems. In addition, in our theory there is a distinguished way of measuring time associated with the foliation of 3D-branes in spacetime. It is called membrane time, see § 5 in Chapter I. By time t in (1.1) and everywhere below we mean one of the possible choices of such membrane time.

The time derivatives of the coordinates of a point particle in (1.1) are components of a vector:

$$v^i = \dot{x}^i = \frac{dx^i}{dt}, \text{ where } i = 1, 2, 3. \tag{1.2}$$

This vector $\mathbf{v}$ in (1.2) is called the particle velocity vector. The acceleration vector $\mathbf{a}$ of a point particle is obtained from the velocity vector $\mathbf{v}$ of this particle through the procedure of covariant differentiation with respect to time:

$$a^i = \nabla_t v^i = \dot{v}^i + \sum_{j=1}^{3}\sum_{k=1}^{3} \Gamma^i_{jk}\, v^i\, v^k, \text{ where } i = 1,\, 2,\, 3. \tag{1.3}$$

Through Γ^i_{jk} in (1.3) we denote the components of the metric connection for the three-dimensional metric g_{ij}, see (2.7) in Chapter II. These quantities are defined by means of the formula (4.1) from Chapter II.

The length of the velocity vector of a point particle $\mathbf{v}$ with the components (1.2) is determined by the metric g_{ij}:

$$|\mathbf{v}| = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, v^i\, v^j}\,. \tag{1.4}$$

Now, given (1.4), we are ready to write the action integral for point particles. When writing the action integral we shall distinguish between baryonic light matter particles and non-baryonic dark matter particles:

$$S_{\text{br}} = -\int m\, c_{\text{br}}^2 \sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\, dt, \tag{1.5}$$

$$S_{\text{nb}} = -\int m\, c_{\text{nb}}^2 \sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}\, dt. \tag{1.6}$$

The integrals (1.5) and (1.6) differ only in the constants c_{br} and c_{nb} in them. These are two of the four speed constants that we considered in (1.2) in Chapter II. In what follows we shall restrict our consideration to the case of non-baryonic matter (1.6). The formulas obtained in this way can be easily adapted to the case of baryonic matter by simply replacing c_{nb} with c_{br} in them.

The action integrals in (1.5) and (1.6) are essentially integrals along the particle's path, which are determined by the functions $x^1(t)$, $x^2(t)$, $x^3(t)$ in (1.1). The quantity g_{00} in these formulas is the scalar function from (2.9) in Chapter II, while the constant m is the mass of the particle, which is also called the rest mass of the particle.

§ 2. Motion of non-baryonic particles in a gravitational field.

The function in the action integral for a point particle is called the Lagrangian of this particle:

$$S_{\text{nb}} = \int L_{\text{nb}}\,dt. \tag{2.1}$$

Comparing (2.1) with (1.6), we obtain

$$L_{\text{nb}} = -m\,c_{\text{nb}}^2\sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}. \tag{2.2}$$

Applying the principle of least action[1] to the integral (2.1) leads to the Euler-Lagrange equations

$$-\frac{d}{dt}\Bigl(\frac{\delta L_{\text{nb}}}{\delta v^i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{x}}} + \Bigl(\frac{\delta L_{\text{nb}}}{\delta x^i}\Bigr)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{v}}} = 0. \tag{2.3}$$

Since the Lagrangian (2.2) is not an integral but a function, the partial variational derivatives are reduced to regular partial derivatives, while the Euler-Lagrange equations themselves (2.3) are written in the form

$$-\frac{d}{dt}\Bigl(\frac{\partial L_{\text{nb}}}{\partial v^i}\Bigr) + \frac{\partial L_{\text{nb}}}{\partial x^i} = 0. \tag{2.4}$$

[1] The principle of least action would be more correctly called the principle of stationary action, since minimal action is never actually required.

The partial derivatives in (2.4) are easily calculated with the use of the formula (2.2). Indeed, we have

$$\begin{gathered}\frac{\partial L_{\text{nb}}}{\partial v^i}=\frac{m\,v_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}},\\ \frac{\partial L_{\text{nb}}}{\partial x^i}=\frac{\displaystyle\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{m}{2}\,\frac{\partial g_{rs}}{\partial x^i}\,v^r\,v^s-c_{\text{nb}}^2\,\frac{m}{2}\,\frac{\partial g_{00}}{\partial x^i}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}.\end{gathered}\tag{2.5}$$

The function g_{00} is a scalar function. Therefore its partial derivative in (2.5) is equal to its covariant derivative:

$$\frac{\partial g_{00}}{\partial x^i}=\nabla_i\,g_{00}.\tag{2.6}$$

It is known that the covariant derivative of a Riemannian metric with respect to its own metric connection is zero: $\nabla_i\,g_{rs}=0$. The formula (4.1) in Chapter II is usually derived from this equality. It is also known that the covariant derivative $\nabla_i\,g_{rs}$ is calculated by means of the following formula (see § 7 in Chapter III of [53]):

$$\nabla_i\,g_{rs}=\frac{\partial g_{rs}}{\partial x^i}-\sum_{q=1}^{3}\Gamma^q_{ir}\,g_{qs}-\sum_{q=1}^{3}\Gamma^q_{is}\,g_{rq}.\tag{2.7}$$

From $\nabla_i\,g_{rs}=0$ and from the formula (2.7) we derive

$$\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\partial g_{rs}}{\partial x^i}\,v^r\,v^s=\sum_{q=1}^{3}\sum_{s=1}^{3}2\,\Gamma^q_{is}\,v_q\,v^s.\tag{2.8}$$

Here Γ^q_{is} are the components of the metric connection defined by the three-dimensional metric g_{ij}. Due to (2.6) and (2.8), the formulas (2.5) are written in the following way:

$$\left.\begin{aligned} \frac{\partial L_{\text{nb}}}{\partial v^i} &= \frac{m\, v_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,,\\ \frac{\partial L_{\text{nb}}}{\partial x^i} &= \frac{\displaystyle\sum_{q=1}^{3}\sum_{s=1}^{3} m\,\Gamma^q_{is}\, v_q\, v^s - \frac{m\, c_{\text{nb}}^2}{2}\,\nabla_i\, g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \end{aligned}\right. \tag{2.9}$$

Applying (2.9) to the equations (2.4), we derive

$$\begin{gathered} \frac{d}{dt}\left(\frac{m\, v_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\right) = \\ = \frac{\displaystyle\sum_{q=1}^{3}\sum_{s=1}^{3} m\,\Gamma^q_{is}\, v_q\, v^s - \frac{m\, c_{\text{nb}}^2}{2}\,\nabla_i\, g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \end{gathered} \tag{2.10}$$

The time derivative in the left hand side of the equation (2.10) is transformed as follows:

$$\begin{gathered} \frac{d}{dt}\left(\frac{m\, v_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\right) = \frac{m\, \dot v_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}} + \\ + \frac{m\, v_i}{\left(\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}\right)^3}\,\frac{1}{2}\left(\frac{d}{dt}\left(\frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}\right) - \dot g_{00} - \sum_{s=1}^{3} v^s\,\nabla_s\, g_{00}\right). \end{gathered} \tag{2.11}$$

By combining (2.10) and (2.11) we derive the differential equations for the components v_i of the velocity vector $\mathbf{v}$:

$$
\begin{gathered}
\frac{m\,\dot v_i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}+\frac{m\,v_i}{\left(\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}\,\right)^3}\,\frac{1}{2}\left(\frac{d}{dt}\left(\frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}\right)-\dot g_{00}-\right.\\
\left.-\sum_{s=1}^{3}v^s\,\nabla_s\,g_{00}\right)=\frac{\displaystyle\sum_{q=1}^{3}\sum_{s=1}^{3}m\,\Gamma^q_{is}\,v_q\,v^s-\frac{m\,c_{\text{nb}}^2}{2}\,\nabla_i\,g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}.
\end{gathered}\tag{2.12}
$$

The equation (2.12) is derived using (2.6). In addition to (2.6) we need the relationship (6.1) from Chapter III. The second term in the left hand side of the equation (2.12) contains the time derivative of $|\mathbf{v}|^2$. We calculate this derivative as follows:

$$
\begin{gathered}
\frac{d(|\mathbf{v}|^2)}{dt}=\frac{d}{dt}\left(\sum_{r=1}^{3}\sum_{s=1}^{3}g_{rs}\,v^r\,v^s\right)=\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{d(\,g_{rs}\,v^r)}{dt}\,v^s+\\
+\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{d(\,g_{rs}\,v^s)}{dt}\,v^r-\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{dg_{rs}}{dt}\,v^s\,v^r=\\
=\sum_{s=1}^{3}2\,\dot v_s\,v^s-\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\partial g_{rs}}{\partial t}\,v^s\,v^r-\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{i=1}^{3}\frac{\partial g_{rs}}{\partial x^i}\,\dot x^i\;v^s\,v^r.
\end{gathered}\tag{2.13}
$$

We transform (2.13) using the formula (6.1) from Chapter III, as well as the formulas (1.2) and (2.8). This gives

$$
\begin{gathered}
\frac{d(|\mathbf{v}|^2)}{dt}=-\sum_{q=1}^{3}\sum_{s=1}^{3}\sum_{i=1}^{3}2\,\Gamma^q_{is}\,v_q\,v^s\,v^i+\\
+\sum_{s=1}^{3}2\,\dot v_s\,v^s-\sum_{r=1}^{3}\sum_{s=1}^{3}2\,c_{\text{gr}}\,b_{rs}\,v^s\,v^r.
\end{gathered}\tag{2.14}
$$

Next we multiply (2.12) by v^i and sum over i from 1 to 3:

$$\frac{m\sum_{i=1}^{3}\dot v_i\,v^i}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}+\frac{m\,|\mathbf{v}|^2}{\left(\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}\,\right)^3}\,\cdot$$

$$\cdot\,\frac{1}{2}\left(\frac{d}{dt}\left(\frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}\right)-\dot g_{00}-\sum_{s=1}^{3}v^s\,\nabla_s\,g_{00}\right)= \tag{2.15}$$

$$=\frac{\sum_{q=1}^{3}\sum_{s=1}^{3}\sum_{i=1}^{3}m\,\Gamma^q_{is}\,v_q\,v^s\,v^i-\dfrac{m\,c_{\text{nb}}^2}{2}\sum_{i=1}^{3}v^i\,\nabla_i\,g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,.$$

We then apply the formula (2.14) to the time derivative of $|\mathbf{v}|^2$ in the formula (2.15). As a result the equality (2.15) simplifies and reduces to the following equality:

$$\sum_{i=1}^{3}\dot v_i\,v^i=\sum_{q=1}^{3}\sum_{s=1}^{3}\sum_{i=1}^{3}\Gamma^q_{is}\,v_q\,v^s\,v^i+\frac{c_{\text{gr}}\,|\mathbf{v}|^2}{g_{00}\,c_{\text{nb}}^2}\sum_{r=1}^{3}\sum_{s=1}^{3}b_{rs}\,\cdot$$

$$\cdot\,v^s\,v^r+\frac{\dot g_{00}\,|\mathbf{v}|^2}{2\,g_{00}}-\frac{c_{\text{nb}}^2}{2}\sum_{i=1}^{3}v^i\,\nabla_i\,g_{00}+\frac{|\mathbf{v}^2|}{g_{00}}\sum_{i=1}^{3}v^i\,\nabla_i\,g_{00}. \tag{2.16}$$

Now we substitute (2.16) into the second term in the right hand side of (2.14). As a result, we get

$$\frac{d(|\mathbf{v}|^2)}{dt}=-\frac{2\,c_{\text{gr}}}{g_{00}}\left(g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}\right)\sum_{r=1}^{3}\sum_{s=1}^{3}b_{rs}\,v^s\,v^r+$$

$$+\,\frac{\dot g_{00}\,|\mathbf{v}|^2}{g_{00}}-c_{\text{nb}}^2\sum_{i=1}^{3}v^i\,\nabla_i\,g_{00}+\frac{2\,|\mathbf{v}|^2}{g_{00}}\sum_{i=1}^{3}v^i\,\nabla_i\,g_{00}. \tag{2.17}$$

The next step is to substitute the formula (2.17) into the equation (2.12). This yields the following equation:

$$
\begin{gathered}
\dot{v}_i-\sum_{q=1}^{3}\sum_{s=1}^{3}\Gamma^q_{is}\,v_q\,\dot{x}^s=-\frac{c_{\text{nb}}^2}{2}\,\nabla_i\,g_{00}+\\
+\,v_i\left(\,\sum_{s=1}^{3}\frac{v^s\,\nabla_s\,g_{00}}{g_{00}}+\frac{\dot{g}_{00}}{2\,g_{00}}+\frac{c_{\text{gr}}}{c_{\text{nb}}^2\,g_{00}}\sum_{r=1}^{3}\sum_{s=1}^{3}b_{rs}\,v^s\,v^r\right).
\end{gathered}
\tag{2.18}
$$

The left hand side of equation (2.18) fits the definition of the covariant derivative of a covector field with respect to parameter t along a parametric curve, see (8.10) in § 8 of Chapter III in [53]). Therefore we can write (2.18) as

$$
\begin{gathered}
\nabla_t v_i=-\frac{c_{\text{nb}}^2}{2}\,\nabla_i\,g_{00}+v_i\left(\,\sum_{s=1}^{3}\frac{v^s\,\nabla_s\,g_{00}}{g_{00}}+\right.\\
\left.+\,\frac{\dot{g}_{00}}{2\,g_{00}}+\frac{c_{\text{gr}}}{c_{\text{nb}}^2\,g_{00}}\sum_{r=1}^{3}\sum_{s=1}^{3}b_{rs}\,v^s\,v^r\right).
\end{gathered}
\tag{2.19}
$$

Note that the equalities (1.2) can be written as differential equations for the particle coordinates:

$$\dot{x}^i=v^i. \tag{2.20}$$

The equations (2.19) supplemented by the equations (2.20) form a complete system of differential equations describing the motion of a non-baryonic particle in a gravitational field defined by a three-dimensional Riemannian metric g_{ij} and a scalar function g_{00}. The equations do not contain the particle mass m. This circumstance is interpreted as the following theorem.

THEOREM 2.1. *The inertial and passive gravitational masses of a non-baryonic massive particle are equal to each other.*

The definitions of inertial, as well as active and passive gravitational masses, are given in [62].

§ 3. Energy and momentum of non-baryonic point particles.

The Legendre transformation for a non-baryonic point particle is determined by its Lagrangian:

$$p_i = \left(\frac{\delta L_{\text{nb}}}{\delta v^i}\right)_{\substack{g,b,\mathbf{g}\\ \mathbf{b},\mathbf{x}}}, \tag{3.1}$$

The Lagrangian (2.2) is not an integral, but a function. Therefore the variational partial derivative in the formula (3.1) should be replaced by a regular partial derivative:

$$p_i = \frac{\partial L_{\text{nb}}}{\partial v^i} \tag{3.2}$$

The partial derivative from (3.2) was already calculated in (2.5). Using the first formula from (2.5), we get

$$p_i = \frac{m\,v_i}{\sqrt{g_{00} - \dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \tag{3.3}$$

The quantities p_i in (3.3) are components of the covector $\mathbf{p}$. This covector with the components (3.3) is called the momentum covector of a non-baryonic particle.

Let's return to the equations (2.10). Using the components of the momentum covector $\mathbf{p}$ from (3.3) and taking into account the equations (1.2), we can write the equations (2.10) as follows:

$$\dot p_i - \sum_{q=1}^{3}\sum_{s=1}^{3}\Gamma^q_{is}\,p_q\,\dot x^s = -\frac{m\,c_{\text{nb}}^2\,\nabla_i\,g_{00}}{2\sqrt{g_{00} - \dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \tag{3.4}$$

The left hand side of the equations (3.4) fits the definition of the covariant derivative of a covector field with respect to parameter

t along a parametric curve, see (8.10) in § 8 of Chapter III in [53]). Therefore, we can write (3.4) as

$$\nabla_t p_i = -\frac{m\,c_{\text{nb}}^2\,\nabla_i\,g_{00}}{2\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \tag{3.5}$$

The equations (3.5) supplemented by the equations (2.20) constitute a complete system of ordinary differential equations describing the motion of a non-baryonic particle in a gravitational field defined by a three-dimensional metric g_{ij} and a scalar function g_{00}. The right hand sides of the equations (3.5) are interpreted as components of the force covector $\mathbf{F}$ which acts upon a non-baryonic particle from the gravitational field:

$$F_i = -\frac{m\,c_{\text{nb}}^2\,\nabla_i\,g_{00}}{2\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \tag{3.6}$$

The energy function for a non-baryonic particle is written in terms of the components of its momentum covector $\mathbf{p}$ and in terms of the components of its velocity vector $\mathbf{v}$:

$$E_{\text{nb}} = \sum_{i=1}^{3} p_i\,v^i - L_{\text{nb}}. \tag{3.7}$$

Applying (3.3) and (2.2) to (3.7), we obtain

$$E_{\text{nb}} = \frac{m\,|\mathbf{v}|^2}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}} + m\,c_{\text{nb}}^2\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}. \tag{3.8}$$

The formula (3.8) for the energy of a non-baryonic particle can be simplified by reducing it to a common denominator and grouping similar terms in the numerator. After simplification the formula (3.8) is written in the following way:

$$E_{\text{nb}} = \frac{m\,c_{\text{nb}}^2}{\sqrt{g_{00} - \dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \tag{3.9}$$

The value of the function g_{00} can be made equal to one at any single point in space, but in general not everywhere. In order to do this it is necessary to change the membrane time according to the formula (2.10) from Chapter II. After this the formula (3.9) takes the following form:

$$E_{\text{nb}} = \frac{m\,c_{\text{nb}}^2}{\sqrt{1 - \dfrac{|\mathbf{v}|^2}{c_{\text{nb}}^2}}}\,. \tag{3.10}$$

Due to (3.10) the constant c_{nb} is interpreted as the upper bound for the speed of a non-baryonic particle.

§ 4. Circular rotation of non-baryonic particles around a Schwarzschild black hole.

In § 8 of Chapter II above we studied Schwarzschild black holes in the framework of our new theory of gravity. For this purpose we used the spacial coordinates

$$x^1 = \rho, \qquad x^2 = \theta, \qquad x^3 = \phi \tag{4.1}$$

and the time variable t associated with a comoving observer placed at infinity from the black hole. The gravitational field of a Schwarzschild black hole is given by the scalar field g_{00} and the three-dimensional diagonal metric g_{ij} which are defined by the formulas (8.7) and (8.6) from Chapter II. The nonzero components of the three-dimensional metric connection are given by the formulas (8.8) from Chapter II.

The function g_{00} and the three-dimensional metric g_{ij} in the case of a Schwarzschild black hole are stationary. They do not depend on time. Therefore we have the following relationships:

$$\dot{g}_{00} = 0, \qquad\qquad b_{ij} = 0 \ \text{ for } \ 1 \leqslant i, j \leqslant 3. \tag{4.2}$$

Let a non-baryonic particle of mass m rotate around a black hole in its equatorial plane with angular velocity ω. Then its rotation in the coordinates (4.1) is given by the formulas

$$\rho(t) = \rho = \text{const}, \qquad \theta(t) = \frac{\pi}{2} = \text{const}, \qquad \phi(t) = \omega\, t. \tag{4.3}$$

Differentiating (4.3) with respect to time, we find the components of the velocity vector of a non-baryonic particle:

$$v^1 = 0, \qquad\qquad v^2 = 0, \qquad\qquad v^3 = \omega. \tag{4.4}$$

The components of the velocity covector are derived from (4.4) using the standard index lowering procedure:

$$v_i = \sum_{k=1}^{3} g_{ik}\, v^k. \tag{4.5}$$

Applying the formulas (8.6) from Chapter II and the formula (4.4) to (4.5) and taking into account (4.3), we obtain

$$v_1 = 0, \qquad\qquad v_2 = 0, \qquad\qquad v_3 = \rho^2\, \omega. \tag{4.6}$$

The components of the acceleration covector are defined as

$$a_i = \nabla_t v_i = \dot{v}_i - \sum_{q=1}^{3} \sum_{s=1}^{3} \Gamma^q_{is}\, v_q\, v^s. \tag{4.7}$$

Applying (4.4) and (4.6) to (4.7) and taking into account the formulas (8.8) from Chapter II, we find

$$a_1 = -\rho\,\omega^2, \qquad a_2 = 0, \qquad a_3 = 0. \tag{4.8}$$

Now we can apply the differential equations (2.19) describing the particle dynamics. First we calculate the components of the gradient for the scalar field g_{00} in them:

$$\nabla_1 g_{00} = \frac{r_{\text{gr}}}{\rho^2} \qquad \nabla_2 g_{00} = 0, \qquad \nabla_3 g_{00} = 0. \tag{4.9}$$

Using (4.4) and (4.9), we derive

$$\sum_{s=1}^{3} v^s\,\nabla_s\,g_{00} = 0. \tag{4.10}$$

Due to (4.2), (4.10), and (4.7) the equation (2.19) reduces to

$$a_i = -\frac{c_{\text{nb}}^2}{2}\,\nabla_i\,g_{00}. \tag{4.11}$$

Applying (4.8) and (4.9) to (4.11), we obtain the equality

$$-\rho\,\omega^2 = -\frac{c_{\text{nb}}^2\,r_{\text{gr}}}{2\,\rho^2}. \tag{4.12}$$

The gravitational radius for a baryonic Schwarzschild black hole of mass M is given by the formula

$$r_{\text{gr}} = \frac{2\,\gamma\,M}{c_{\text{gr}}^2}. \tag{4.13}$$

The formula (4.13) is contained in § 100 of Chapter XII in [2] and in [63]. Substituting (4.13) into (4.12), we derive

$$\rho\,\omega^2 = \frac{c_{\text{nb}}^2\,\gamma\,M}{c_{\text{gr}}^2\,\rho^2}. \tag{4.14}$$

Despite the presence of a denominator in the formula for the force (3.6), the formula (4.14) coincides with the corresponding formula from the classical Newtonian theory of gravity up to the constant scalar factor

$$k_{\text{nb}} = \frac{c_{\text{nb}}^2}{c_{\text{gr}}^2}. \tag{4.15}$$

If we replace the non-baryonic particle with a baryonic one, then c_{nb} in (4.15) will be replaced by c_{br}. The corresponding factor k_{br} in this case must be equal to one, since for baryonic matter in the classical Newtonian theory of gravity there is no speed of light and its analogues:

$$k_{\text{br}} = \frac{c_{\text{br}}^2}{c_{\text{gr}}^2} = 1. \tag{4.16}$$

From (4.16) follows the equality

$$c_{\text{br}} = c_{\text{gr}}, \tag{4.17}$$

while the above reasoning serves as a proof of the obtained equality (4.17).

§ 5. Superbradyons of Luis Gonzalez-Mestres.

Massive non-baryonic particles with the limiting speed different from the speed of light were considered by Luis Gonzalez-Mestres in [64]. He believed that

$$c_{\text{nb}} > c_{\text{el}}. \tag{5.1}$$

Due to the inequality (5.1), Luis Gonzalez-Mestres in [65] called the particles he invented superbradyons. In [66] and [67] he wrote a formula for the energy of superbradyons of the form (3.10) and suggested that superbradyons could be found in cosmic rays.

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APPENDIX.

List of the author's publications for the period from 1986 to 2025.

Part 1. Theory of solitons.

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