BOUNDARY CONDITIONS IN LAGRANGIAN FIELD THEORIES.

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Abstract. Deriving field equations in Lagrangian field theories is based on the stationary action principle. Typically this principle is applied to smooth and continuous configurations of fields. Assuming that the stationary action principle remains valid for discontinuous configurations of fields, we derive boundary value conditions for fields in Lagrangian field theories.

1. Introduction.

Most of the Lagrangian field theories nowadays are four-dimensional relativistic theories describing the universe in terms of the four-dimensional spacetime. A different paradigm is suggested in the 3D-brane universe model. The 3D-brane universe model is the name of a new non-Einsteinian theory of gravity where the spacetime is treated as just a mathematical abstraction that records various moments in the evolution of the real three-dimensional universe (see the book [1] or its English translation [2], see also the papers [3–16] and the conference report abstracts [17–30] given in chronological order). The main goal of the present paper is to prepare a background for studying matter fields within the new theory. For this reason here we consider three-dimensional field theories with the time evolution and borrow some notations from [1] and [2].

2. Comoving coordinates and membrane time.

Let M be our universe. According to the paradigm from [1] and [2] it is a three-dimensional Riemannian manifold with the time dependent metric

$$g_{ij} = g_{ij}(t, x^1, x^2, x^3). \tag{2.1}$$

At each moment of time t we have some definite special embedding

$$f_t : M \longrightarrow M_4 \tag{2.2}$$

of the real physical universe M into the four-dimensional pseudo-Riemannian manifold M_4 which is called the spacetime. According to the paradigm from [1] and [2] the spacetime M_4 in (2.2) is not a physical entity, but a mathematical abstraction

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only. Despite this, we keep the concept of spacetime as a bridge connecting the new theory with Einstein's general relativity.

The images $f_t(M)$ of M in the spacetime M_4 are 3D-branes. They are three-dimensional space-like submanifolds of M_4. These branes do not intersect with each other and constitute a foliation of 3D-branes in the spacetime. This foliation of 3D-branes is the fourth geometric structure arising in the new theory and being complementary to the first three structures known in Einstein's general relativity[1]:

1) a pseudo-Riemannian metric $\mathbf{G}$ with the signature $(+,-.-,-)$;
2) an orientation;
3) a polarization;
4) a foliation of spacelike 3D-branes filling the spacetime entirely with the exception of perhaps one point corresponding to the Big Bang.

The variables x^1, x^2, x^3 in (2.1) are some dedicated local coordinates in M, which are called *comoving coordinates* (see § 3 of Chapter I in [2]). The mappings (2.2) associate these coordinates with the *spacial comoving coordinates* x^1, x^2, x^3 in the spacetime M_4 (see definitions and more details in § 3 of Chapter I in [2]).

Definition 4.1. Observers whose comoving coordinates do not change over time are called comoving observers.

Comoving observers are considered to be in the state of absolute rest (see § 4 of Chapter I in [2]). Comoving coordinates in M_4 are dedicated coordinates. The variable t in (2.2) is also associated with a dedicated time variable in M_4. It is called the *membrane time*. Using it, one can introduce the fourth coordinate x^0 in M_4 complementary to the spacial comoving coordinates x^1, x^2, x^3:

$$x^0 = c_{\text{gr}}\, t. \tag{2.3}$$

The constant c_{gr} in (2.3) is interpreted as the *speed of gravity* or (more exactly) the *speed of gravitational waves* (see details in § 1 of Chapter II in [2]).

In comoving spacial coordinates x^1, x^2, x^3 complemented with the temporal coordinate x^0 from (2.3) the pseudo-Riemannian metric $\mathbf{G}$ of the spacetime M_4 is given by the following block-diagonal matrix:

$$G_{ij} = \begin{Vmatrix} g_{00} & 0 & 0 & 0 \\ 0 & -g_{11} & -g_{12} & -g_{13} \\ 0 & -g_{21} & -g_{22} & -g_{23} \\ 0 & -g_{31} & -g_{32} & -g_{33} \end{Vmatrix}, \tag{2.4}$$

see § 2 of Chapter II in [2]. The components of the lower right diagonal block of the matrix are determined by the components of the three-dimensional metric (2.1). The component g_{00} in (2.4) in the new theory is interpreted as a scalar function:

$$g_{00} = g_{00}(t, x^1, x^2, x^3). \tag{2.5}$$

The functions (2.1) and (2.5) are dynamic variables describing the gravitational field in the new theory, see § 2 of Chapter II in [2].

[1] See more details in § 2 of Chapter I in [2] and in § 3 of Chapter III in [31].

3. Action integrals and Lagrangians.

Action integrals in field theories are usually written as time integrals of Lagrangians, while Lagrangians are spacial integrals of Lagrangian densities:

$$S=\int L\,dt, \qquad\qquad L=\int \mathcal{L}\,\sqrt{\det g}\;d^3x. \tag{3.1}$$

The Lagrangian L in (3.1) depends on the function (2.5) and on its time derivative

$$\dot g_{00}(t,x^1,x^2,x^3)=\frac{\partial g_{00}}{\partial t}. \tag{3.2}$$

Along with g_{00} and $\dot g_{00}$, the Lagrangian L in (3.1) depends on the functions (2.1) and on their time derivatives. Following § 4 from Chapter II in [2], we replace the time derivatives of the functions (2.1) by the functions

$$b_{ij}(t,x^1,x^2,x^3)=\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}=\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial g_{ij}}{\partial t}=\frac{\dot g_{ij}}{2\,c_{\text{gr}}}. \tag{3.3}$$

The functions (2.1), (3.3), (2.5), and (3.2) correspond to the gravitational field. Apart from the gravitational field, the theory includes matter. Like in § 3 from Chapter III in [2], we do not specify any particular sort of matter. In general form we assume that matter is described by some functions

$$Q_i=Q_i(t,x^1,x^2,x^3),\quad 1\leqslant i\leqslant n, \tag{3.4}$$

and their time derivatives

$$\dot Q_i(t,x^1,x^2,x^3)=\frac{\partial Q_i}{\partial t},\quad 1\leqslant i\leqslant n. \tag{3.5}$$

Therefore in symbolic form the total set of functional arguments of the Lagrangian L is presented by the following formula:

$$L=L(g,\dot g,\mathbf{g},\mathbf{b},\mathbf{Q},\dot{\mathbf{Q}}). \tag{3.6}$$

The arguments g and $\dot g$ in (3.6) correspond to the functions (2.5) and (3.2), the arguments $\mathbf{g}$ and $\mathbf{b}$ correspond to the functions (2.1) and (3.3), and the arguments $\mathbf{Q}$ and $\dot{\mathbf{Q}}$ correspond to the functions (3.4) and (3.5). For the Lagrangian density $\mathcal{L}$ the equality (3.6) means that $\mathcal{L}$ depends not only on the values of the functions (2.5), (3.2), (2.1), (3.3), (3.4), and (3.5), but also on the values of some finite number of their partial derivatives of various orders with respect to x^1, x^2, x^3.

4. Matter boundaries and interfaces.

Some sorts of matter in the universe may have clear and sharp boundaries, e. g. rock to vacuum interface of asteroids or ice to water interface of icebergs. In order to describe such interfaces we consider a two-dimensional surface σ in the three-dimensional real physical universe M. This surface can move, e. g. because of the translational and rotational motion of an asteroid or because of melting of an iceberg in a hot sea. Let's denote through $\mathbf{u}=\mathbf{u}(P)$ the velocity vector of a point

$P \in \sigma$. The surface σ divides the three-dimensional universe M into two parts M^+ and M^- as shown in Fig. 4.1.

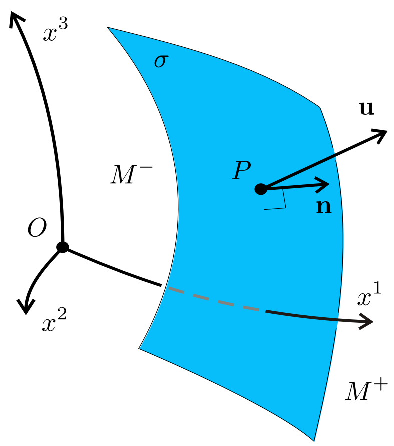

Fig. 4.1

Through $\mathbf{n} = \mathbf{n}(P)$ we denote the unit normal vector to the surface σ at a point $P \in \sigma$. The choice of this vector fixes one of the two possible orientations on the surface σ. Taking into account this orientation, we can write

$$\begin{aligned} &\sigma = \partial M^-, \\ &\sigma = -\,\partial M^+. \end{aligned} \tag{4.1}$$

Previously, developing the new theory in [3–16] we considered smooth Lagrangian densities $\mathcal{L}$ in (3.1) and smooth configurations of the fields (2.1), (2.5), (3.4) and their time derivatives (3.2), (3.3), and (3.5). Here in the presence of matter interfaces we proceed to discontinuous Lagrangian densities $\mathcal{L}$ in (3.1). We define them as follows:

$$\mathcal{L} = \begin{cases} \mathcal{L}^- & \text{if } \ P \in M^-, \\ \mathcal{L}^+ & \text{if } \ P \in M^+. \end{cases} \tag{4.2}$$

The lists of arguments of both functions $\mathcal{L}^-$ and $\mathcal{L}^+$ in (4.2) are assumed to be the same, see (3.6) and the text below this formula. Through P in (4.2) we denote a point of the real three-dimensional universe M whose comoving coordinates are x^1, x^2, x^3. Since the surface σ is a moving surface, the domains M^- and M^+ in (4.2) depend on the time variable t used in the arguments of the functions (2.1), (2.5), (3.4), (3.2), (3.3), and (3.5), which is the global time in M associated with the membrane time in M_4.

5. Stationary action principle and variations of the action integral.

The stationary action principle (see [32]) states that the variation of an action integral with respect to each dynamical variable should be zero. In our case the action integral S is given by the formulas (3.1). Let's choose the function g_{00} from (2.5) as the first dynamical variable. In the case of a smooth Lagrangian density $\mathcal{L}$ the stationary action principle applied to S with respect to the dynamical variable g_{00} yields the following differential equation:

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} - c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = 0, \tag{5.1}$$

see § 3 of Chapter III in [2]. The equation (5.1) remains valid for the Lagrangian density (4.2) within the bulk of the domains M^- and M^+, i.e. we have

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} - c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}^-}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}} = 0 \quad \text{for} \quad P \in M^- \tag{5.2}$$

and we have a similar equation

$$-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{L}^{+}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-c_{\text{gr}}\,\Bigl(\frac{\delta\mathcal{L}^{+}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sum_{q=1}^{3}b^{q}_{q}+\Bigl(\frac{\delta\mathcal{L}^{+}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}=0\quad\text{for}\quad P\in M^{+}. \tag{5.3}$$

Our goal below is to apply the stationary action principle to the action integral S in (3.1) with respect to the dynamic variable g_{00} at a point $P \in \sigma$, where $\sigma = \partial M^- = -\,\partial M^+$ (see (4.1) and Fig. 4.1). For this purpose we consider the following small variation of the function g_{00}:

$$\hat g_{00}=g_{00}(t,x^1,x^2,x^3)+\varepsilon\,h_{00}(t,x^1,x^2,x^3). \tag{5.4}$$

Here $\varepsilon \to 0$ is a small parameter, while $h(t,x^1,x^2,x^3)$ is an arbitrary smooth function with compact support (see [33]). Differentiating (5.4) with respect to the time variable t, we get the following equality:

$$\dot{\hat g}_{00}=\dot g_{00}(t,x^1,x^2,x^3)+\varepsilon\,\dot h_{00}(t,x^1,x^2,x^3). \tag{5.5}$$

Due to (4.2) the Lagrangian L in (3.1) is transformed to the sum of two integrals:

$$L=\int\limits_{M^-}\mathcal{L}^{-}\,\sqrt{\det g}\;d^3x+\int\limits_{M^+}\mathcal{L}^{+}\,\sqrt{\det g}\;d^3x. \tag{5.6}$$

Now, like in § 9 of Chapter III in [2], we introduce the following notations:

$$g_{00}[i_1\dots i_s]=\frac{\partial g_{00}}{\partial x^{i_1}\dots\partial x^{i_s}},\qquad\qquad h_{00}[i_1\dots i_s]=\frac{\partial h_{00}}{\partial x^{i_1}\dots\partial x^{i_s}}, \tag{5.7}$$

$$\dot g_{00}[i_1\dots i_s]=\frac{\partial \dot g_{00}}{\partial x^{i_1}\dots\partial x^{i_s}},\qquad\qquad \dot h_{00}[i_1\dots i_s]=\frac{\partial \dot h_{00}}{\partial x^{i_1}\dots\partial x^{i_s}}. \tag{5.8}$$

Each higher order partial derivative of the functions g_{00}, h_{00}, $\dot g_{00}$, $\dot h_{00}$ with respect to spacial coordinates x^1, x^2, x^3 can be expressed in the form of (5.7) and (5.8). In order to exclude duplicates we implicitly assume that

$$1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_s\leqslant 3. \tag{5.9}$$

Substituting (5.4) and (5.5) for g_{00} and $\dot g_{00}$ into the integrals (5.6) and expanding them with respect to the small parameter $\varepsilon \to 0$, we get

$$\hat L=L+\varepsilon\,L_1+\ldots\,. \tag{5.10}$$

By ellipsis in (5.10) we denote higher order terms with respect to the small parameter $\varepsilon \to 0$. The first order term L_1 in (5.10) is subdivided into two parts

$$L_1=L_1^{-}+L_1^{+}. \tag{5.11}$$

Each derivative $g_{00}[i_1\dots i_s]$ of the from (5.7) makes its own contributions to L_1^- and to L_1^+. Let's denote these contributions through $I_g^-[i_1\dots i_s]$ and $I_g^+[i_1\dots i_s]$.

In explicit form these contributions are given by the following integrals:

$$
\begin{aligned}
I_g^-[i_1\dots i_s]&=\int\limits_{M^-}\Bigl(\frac{\partial\mathcal{L}^-}{\partial g_{00}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\dots i_s]\,d^3x,\\
I_g^+[i_1\dots i_s]&=\int\limits_{M^+}\Bigl(\frac{\partial\mathcal{L}^+}{\partial g_{00}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\dots i_s]\,d^3x.
\end{aligned}
\tag{5.12}
$$

Each derivative $\dot g_{00}[i_1\dots i_s]$ of the from (5.8) also makes its own contributions to L_1^- and to L_1^+ in (5.11). We denote these contributions through $I_{\dot g}^-[i_1\dots i_s]$ and $I_{\dot g}^+[i_1\dots i_s]$. In explicit form these contributions are given by the integrals

$$
\begin{aligned}
I_{\dot g}^-[i_1\dots i_s]&=\int\limits_{M^-}\Bigl(\frac{\partial\mathcal{L}^-}{\partial\dot g_{00}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\dots i_s]\,d^3x,\\
I_{\dot g}^+[i_1\dots i_s]&=\int\limits_{M^+}\Bigl(\frac{\partial\mathcal{L}^+}{\partial\dot g_{00}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\dots i_s]\,d^3x.
\end{aligned}
\tag{5.13}
$$

Like in § 9 of Chapter III in [2], here we denote through ι_q a linear mapping acting upon differential 3-forms and generating differential 2-forms so that

$$
\iota_q(dx^1\wedge dx^2\wedge dx^3)=\begin{cases}dx^2\wedge dx^3 & \text{if}\ \ q=1,\\ dx^3\wedge dx^1 & \text{if}\ \ q=2,\\ dx^1\wedge dx^2 & \text{if}\ \ q=3.\end{cases}
\tag{5.14}
$$

Applying (5.14) and acting as in deriving the formula (9.7) in § 9 of Chapter III in [2], we derive the following formula for the first integral in (5.12):

$$
\begin{aligned}
I_g^-[i_1\dots i_s]&=\sum_{r=1}^{s}\int\limits_{\sigma}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^-}{\partial g_{00}[i_1\dots i_s]}\,\cdot\\
&\quad\cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\dots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\
&+\int\limits_{M^-}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^-}{\partial g_{00}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{00}\,d^3x.
\end{aligned}
\tag{5.15}
$$

The second integral (5.12) is transformed similarly:

$$
\begin{aligned}
I_g^+[i_1\dots i_s]&=\sum_{r=1}^{s}\int\limits_{\sigma}(-1)^{r}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^+}{\partial g_{00}[i_1\dots i_s]}\,\cdot\\
&\quad\cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\dots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\
&+\int\limits_{M^+}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\dots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^+}{\partial g_{00}[i_1\dots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{00}\,d^3x.
\end{aligned}
\tag{5.16}
$$

Slight difference in signs in the formulas (5.15) and (5.16) is due to the difference in signs in two formulas (4.1).

Now we proceed to the formulas (5.13). They are similar to (5.12). The first integral in (5.13) is transformed in the following way:

$$\begin{gathered}I^{-}_{\dot g}[i_1\ldots i_s]=\sum^{s}_{r=1}\int\limits_{\sigma}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{-}}{\partial\dot g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\int\limits_{M^{-}}(-1)^{s}\,\frac{\partial^{s}}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{-}}{\partial\dot g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\dot h_{00}\,d^3x.\end{gathered}\tag{5.17}$$

The second integral (5.13) is transformed similarly:

$$\begin{gathered}I^{+}_{\dot g}[i_1\ldots i_s]=\sum^{s}_{r=1}\int\limits_{\sigma}(-1)^{r}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{+}}{\partial\dot g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\int\limits_{M^{+}}(-1)^{s}\,\frac{\partial^{s}}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{+}}{\partial\dot g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\dot h_{00}\,d^3x.\end{gathered}\tag{5.18}$$

Slight difference in signs in the formulas (5.17) and (5.18) is also due to the difference in signs in two formulas (4.1).

The last terms in the formulas (5.15) and (5.16) are related to the bulk of the domains M^- and M^+. They contribute to the variational derivatives

$$\Bigl(\frac{\delta\mathcal{L}^{-}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}, \qquad\qquad \Bigl(\frac{\delta\mathcal{L}^{+}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}.\tag{5.19}$$

The total contribution of all these terms (including those with $s=0$) to the first order term L_1 in the expansion (5.11) is given by the formula

$$I^{\pm}_{g}=\int\limits_{M^{-}}\Bigl(\frac{\delta\mathcal{L}^{-}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}h_{00}\,\sqrt{\det g}\,\,d^3x+\int\limits_{M^{+}}\Bigl(\frac{\delta\mathcal{L}^{+}}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}h_{00}\,\sqrt{\det g}\,\,d^3x\tag{5.20}$$

containing the derivatives (5.19). The last terms in the formulas (5.17) and (5.18) are also related to the bulk of the domains M^- and M^+. They contribute to the following two variational derivatives:

$$\Bigl(\frac{\delta\mathcal{L}^{-}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}, \qquad\qquad \Bigl(\frac{\delta\mathcal{L}^{+}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}.\tag{5.21}$$

In this case we can also find the total contribution of all such terms (including those with $s=0$) to the first order term L_1 in the expansion (5.11). This contribution is given by a formula similar to the formula (5.20):

$$I^{\pm}_{\dot g}=\int\limits_{M^-}\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\dot h_{00}\,\sqrt{\det g}\;d^3x\,+\int\limits_{M^+}\Bigl(\frac{\delta\mathcal{L}^+}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\dot h_{00}\,\sqrt{\det g}\;d^3x \tag{5.22}$$

containing the above derivatives (5.21).

Note that the integrals (5.22) are parts of the first order term L_1 in the expansion (5.11), while L_1 determines the first order term

$$S_1=\int L_1\,dt. \tag{5.23}$$

in the expansion of the action integral

$$\hat S=S+\varepsilon\,S_1+\ldots\,. \tag{5.24}$$

The relationships (5.23) and (5.24) follow from (3.1). Due to (5.23) both integrals (5.22) are subject to subsequent integration over time, i.e. we have

$$\begin{aligned}S^{\pm}_{\dot g}=&\int\int\limits_{M^-}\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\dot h_{00}\,\sqrt{\det g}\;d^3x\,dt\,+\\ &+\int\int\limits_{M^+}\Bigl(\frac{\delta\mathcal{L}^+}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\dot h_{00}\,\sqrt{\det g}\;d^3x\,dt.\end{aligned} \tag{5.25}$$

Integrating by parts in two integrals (5.25), we derive

$$\begin{aligned}S^{\pm}_{\dot g}=&\int\int\limits_{\sigma}(\mathbf{u},\mathbf{n})\biggl(\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\Bigl(\frac{\delta\mathcal{L}^+}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\biggr)\,h_{00}\,dS\,dt\,-\\ &-\int\int\limits_{M^-}\frac{\partial}{\partial t}\biggl(\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sqrt{\det g}\biggr)\,h_{00}\,d^3x\,dt\,-\\ &-\int\int\limits_{M^+}\frac{\partial}{\partial t}\biggl(\Bigl(\frac{\delta\mathcal{L}^+}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sqrt{\det g}\biggr)\,h_{00}\,d^3x\,dt.\end{aligned} \tag{5.26}$$

Here through $(\mathbf{u},\mathbf{n})$ we denote the scalar product of the surface velocity vector $\mathbf{u}$ and its unit normal vector, see Fig. 4.1. Through dS in (5.26) we denote the area element of the surface σ.

Due to (5.23) the integrals (5.20) are also subject to integration over time:

$$\begin{aligned}I^{\pm}_{g}=&\int\int\limits_{M^-}\Bigl(\frac{\delta\mathcal{L}^-}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sqrt{\det g}\;h_{00}\,d^3x\,dt\,+\\ &+\int\int\limits_{M^+}\Bigl(\frac{\delta\mathcal{L}^+}{\delta g_{00}}\Bigr)_{\substack{\dot g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\sqrt{\det g}\;h_{00}\,d^3x\,dt.\end{aligned} \tag{5.27}$$

The last two terms in (5.26) and the integrals (5.27) are related to the bulk of the domains M^- and M^+. Applying the stationary action principle to them leads to the equations (5.2) and (5.3). Therefore we omit last two terms in (5.26), replacing them by ellipsis, and, returning back to the formula (5.22), we write

$$I^{\pm}_{\dot g}=\int\limits_{\sigma}(\mathbf{u},\mathbf{n})\biggl(\Bigl(\frac{\delta\mathcal{L}^{-}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\Bigl(\frac{\delta\mathcal{L}^{+}}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\biggr)\,h_{00}\,dS+\ldots. \tag{5.28}$$

Similarly, we omit the last terms in (5.15), (5.16), (5.17), and (5.18), replacing them by ellipses, since they are already taken into account in the variational derivatives (5.19) and (5.21) and in the differential equations (5.2) and (5.3). Their boundary effect is expressed by the formula (5.28). So, we have

$$\begin{aligned}I^{-}_{g}[i_1\ldots i_s]=\sum_{r=1}^{s}\int\limits_{\sigma}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{-}}{\partial g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\ldots,\end{aligned}\tag{5.29}$$

$$\begin{aligned}I^{+}_{g}[i_1\ldots i_s]=\sum_{r=1}^{s}\int\limits_{\sigma}(-1)^{r}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{+}}{\partial g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\ldots,\end{aligned}\tag{5.30}$$

$$\begin{aligned}I^{-}_{\dot g}[i_1\ldots i_s]=\sum_{r=1}^{s}\int\limits_{\sigma}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{-}}{\partial\dot g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\ldots,\end{aligned}\tag{5.31}$$

$$\begin{aligned}I^{+}_{\dot g}[i_1\ldots i_s]=\sum_{r=1}^{s}\int\limits_{\sigma}(-1)^{r}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}^{+}}{\partial\dot g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\ldots.\end{aligned}\tag{5.32}$$

Due to (5.10) and (5.11) the expressions (5.28), (5.29), (5.30), (5.31), and (5.32) constitute the first order term L_1 in the expansion of the Lagrangian $\hat L$:

$$\begin{aligned}L_1=\sum_{s=1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\bigl(I^{-}_{g}[i_1\ldots i_s]+I^{+}_{g}[i_1\ldots i_s]+\\ +I^{-}_{\dot g}[i_1\ldots i_s]+I^{+}_{\dot g}[i_1\ldots i_s]\bigr)+I^{\pm}_{\dot g}.\end{aligned}\tag{5.33}$$

Here N is a finite integer number being an upper bound for the orders of spacial derivatives of the function g_{00} included in the Lagrangian density $\mathcal{L}$. In order to avoid duplicates the summation indices in (5.33) should obey the inequalities (5.9). Note also that the partial differential operators of the form

$$\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\tag{5.34}$$

in the terms with $r=1$ in (5.29), (5.30), (5.31), and (5.32) should be ignored.

According to (3.1), the expansion (5.10) of the Lagrangian $\hat{L}$ produces the expansion (5.24) of the action integral $\hat{S}$. The first order term S_1 in (5.24) is given by the integral (5.23). Substituting (5.33) into (5.23), we get

$$\begin{aligned}S_1=\sum_{s=1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\Bigl(\int I_g^{-}[i_1\ldots i_s]\,dt+\int I_g^{+}[i_1\ldots i_s]\,dt\,+\\ +\int I_{\dot g}^{-}[i_1\ldots i_s]\,dt+\int I_{\dot g}^{+}[i_1\ldots i_s]\,dt\Bigr)+\int I_{\dot g}^{\pm}\,dt.\end{aligned}\tag{5.35}$$

Let's perform some transformations with the integrands in (5.35). We replace summation over r in (5.29), (5.30), (5.31), and (5.32) with summation over $s-r$. As a result the formula (5.33) is written as

$$\begin{aligned}L_1=\sum_{s=1}^{N}\sum_{r=0}^{s-1}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\sum_{s=1}^{N}\sum_{r=0}^{s-1}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial \dot g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)+I_{\dot g}^{\pm}+\ldots\,.\end{aligned}\tag{5.36}$$

Then we swap sums over s and over r in (5.36). This yields

$$\begin{aligned}L_1=\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\ +\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial \dot g_{00}[i_1\ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)+I_{\dot g}^{\pm}+\ldots\,.\end{aligned}\tag{5.37}$$

Again let's recall that L_1 is the first order term in the expansion (5.10) and that it should be integrated over time in (5.23). When integrating over time the second term of (5.37) we can apply integration by parts:

$$\begin{aligned}&\int\int\limits_{\sigma}\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial \dot g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\dot h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge\\ &\wedge dx^3)\,dt=-\int\int\limits_{\sigma}\frac{\partial}{\partial t}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial \dot g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_r]\,\cdot\\ &\cdot\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)\,dt-\int\int\limits_{\sigma}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial \dot g_{00}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\cdot\end{aligned}$$

$$\cdot(\mathbf{u},\mathbf{n})\,h_{00}[i_1\ldots i_r]\,\frac{dS\,dt}{\sqrt{\det g}}-\int\!\!\int\limits_{\sigma}\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial\dot g_{00}[i_1\ldots i_s]}\sqrt{\det g}\,\Bigr)\cdot$$
$$\cdot(\mathbf{u},\mathbf{n})\,h_{00}[i_1\ldots i_{r+1}]\,\frac{dS\,dt}{\sqrt{\det g}}$$

Like in (5.26) and (5.28), through $(\mathbf{u},\mathbf{n})$ in the above formula we denote the scalar product of the surface velocity vector $\mathbf{u}$ and its unit normal vector, see Fig. 4.1. Through dS in this formula we denote the area element of the surface σ. Applying this formula back to the formula (5.37), we get

$$\begin{aligned}
L_1={}&\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial g_{00}[i_1\ldots i_s]}\cdot\\
&\cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)-\\
&-\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial}{\partial t}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial\dot g_{00}[i_1\ldots i_s]}\cdot\\
&\cdot\sqrt{\det g}\,\Bigr)\,h_{00}[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)-\\
&-\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial\dot g_{00}[i_1\ldots i_s]}\cdot\\
&\cdot\sqrt{\det g}\,\Bigr)\,(\mathbf{u},\mathbf{n})\,h_{00}[i_1\ldots i_r]\,\frac{dS}{\sqrt{\det g}}+\\
&+\sum_{r=1}^{N}\sum_{s=r}^{N}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial\dot g_{00}[i_1\ldots i_s]}\cdot\\
&\cdot\sqrt{\det g}\,\Bigr)\,(\mathbf{u},\mathbf{n})\,h_{00}[i_1\ldots i_r]\,\frac{dS}{\sqrt{\det g}}+\\
&+\int\limits_{\sigma}(\mathbf{u},\mathbf{n})\biggl(\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\Bigl(\frac{\delta\mathcal{L}^+}{\delta\dot g_{00}}\Bigr)_{\substack{g,\mathbf{g},\mathbf{b}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\biggr)\,h_{00}\,dS+\ldots.
\end{aligned}\tag{5.38}$$

Like in the case of (5.34), the partial differential operators of the form

$$\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\qquad\qquad\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\tag{5.39}$$

should be ignored in those terms of the formula (5.38) where $s=r+1$ and $s=r$ respectively. Note that some terms in the third and the fours groups of summands in (5.38) do cancel each other. However not all of them are canceled and taking the cancellations into account will not make the formula much simpler.

Note that in (5.38) we see a mixture of the integrals of the first and the second kind over the surface σ. We can bring all of them to integrals of the second kind using the following formula:

$$\int\limits_{\sigma}(\mathbf{u},\mathbf{n})\,A\,dS=\sum_{k=1}^{3}\int\limits_{\sigma}\sqrt{\det g}\,(\mathbf{u},\mathbf{n})\,A\,n^k\,\iota_k(dx^1\wedge dx^2\wedge dx^3).\tag{5.40}$$

The formula (5.40) is a special case of the more general formula

$$\sum_{k=1}^{3}\int\limits_{\sigma}\sqrt{\det g}\,\mathcal{J}^k\,\iota_k(dx^1\wedge dx^2\wedge dx^3)=\int\limits_{\sigma}(\mathbf{J},\mathbf{n})\,dS. \tag{5.41}$$

Using the above formulas (5.38), (5.40), and (5.41), now we define a series of vector fields $\mathbf{J}_g[i_1\ldots i_r]$ such that the formula

$$L_1=\sum_{r=0}^{N}\sum_{i_1}^{3}\underset{\ldots}{\ldots}\sum_{i_r}^{3}\mathcal{J}_g[i_1\ldots i_r]^k\,h_{00}[i_1\ldots i_r]\,\iota_k(dx^1\wedge dx^2\wedge dx^3)+\ldots. \tag{5.42}$$

coincides with the formula (5.38) for any smooth function with compact support $h_{00}(t,x^1,x^2,x^3)$ in (5.4). Here in (5.42) k is an upper index like in the formulas (5.40) and (5.41). The indices $i_1,\ldots,i_r$ in (5.38) are subordinate to the indices $i_1,\ldots,i_s$. They obey the inequalities

$$1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_r\leqslant 3. \tag{5.43}$$

produced from (5.9). The indices $i_1,\ldots,i_r$ in (5.42) also obey the inequalities (5.43). Due to the formula (5.40) the components $\mathcal{J}_g[i_1\ldots i_r]^k$ of the vector fields $\mathbf{J}_g[i_1\ldots i_r]$ are uniquely determined from (5.38).

Note that the vector fields $\mathbf{J}_g[i_1\ldots i_r]$ are defined not globally in the universe M, but only at the points of the surface σ, though they are not inner vector fields of this surface.

The stationary action principle means that the first order term in the Lagrangian expansion (5.10) with respect to the variation (5.4) of the dynamic variable g_{00} given by the (5.42) should be identically zero. This leads to the boundary conditions

$$\sum_{k=1}^{3}\mathcal{J}_g[i_1\ldots i_r]^k\,n_k\,\Bigr|_{P\in\sigma}=0. \tag{5.44}$$

Theorem 5.1. *If the stationary action principle is valid for discontinuous Lagrangian densities of the form (4.2), then field configurations of corresponding Lagrangian field theories obey the boundary conditions (5.44) on the interface boundary $\sigma=\partial M^-=-\partial M^+$.*

6. Boundary conditions associated with the three-dimensional metric.

The boundary conditions (5.44) are associated with the scalar function (2.5) and its variation (5.4). In the case of the three-dimensional metric (2.1) its small variation is written in the following form:

$$\hat{g}_{ij}=g_{ij}(t,x^1,x^2,x^3)+\varepsilon\,h_{ij}(t,x^1,x^2,x^3). \tag{6.1}$$

Here $\varepsilon\to 0$ is a small parameter, while $h_{ij}(t,x^1,x^2,x^3)$ are arbitrary smooth functions with compact support (see [33]). Differentiating the equality (6.1) with respect to t and taking into account the formula (3.3), we get the following equality:

$$\hat{b}_{ij} = b_{ij}(t, x^1, x^2, x^3) + \frac{\varepsilon}{2\, c_{\text{gr}}}\, \dot{h}_{ij}(t, x^1, x^2, x^3). \tag{6.2}$$

The formula (5.6) remains unchanged, while the formulas (5.7) and (5.8) are replaced with the following formulas:

$$g_{ij}[i_1 \ldots i_s] = \frac{\partial g_{ij}}{\partial x^{i_1} \ldots \partial x^{i_s}}, \qquad h_{ij}[i_1 \ldots i_s] = \frac{\partial h_{ij}}{\partial x^{i_1} \ldots \partial x^{i_s}}, \tag{6.3}$$

$$b_{ij}[i_1 \ldots i_s] = \frac{\partial b_{ij}}{\partial x^{i_1} \ldots \partial x^{i_s}}, \qquad \dot{h}_{ij}[i_1 \ldots i_s] = \frac{\partial \dot{h}_{ij}}{\partial x^{i_1} \ldots \partial x^{i_s}}. \tag{6.4}$$

The variations (6.1) and (6.2) lead to the variations

$$\hat{L} = L + \varepsilon\, L_1 + \ldots, \qquad \hat{S} = S + \varepsilon\, S_1 + \ldots \tag{6.5}$$

similar to (5.10) and (5.24). By ellipses in (6.5) we denote higher order terms with respect to the small parameter $\varepsilon \to 0$. Due to (4.2) and (5.6) the first order term L_1 in (6.5) is subdivided into two parts

$$L_1 = L_1^- + L_1^+. \tag{6.6}$$

Each derivative $g_{ij}[i_1 \ldots i_s]$ of the from (6.3) and each derivative $b_{ij}[i_1 \ldots i_s]$ of the form (6.4) makes its own contributions to L_1^- and to L_1^+ in (6.6). Let's denote these contributions through $I^-_{g_{ij}}[i_1 \ldots i_s]$, $I^+_{g_{ij}}[i_1 \ldots i_s]$, $I^-_{b_{ij}}[i_1 \ldots i_s]$, and $I^+_{b_{ij}}[i_1 \ldots i_s]$. In explicit form these contributions are given by the following integrals:

$$\begin{aligned} I^-_{g_{ij}}[i_1 \ldots i_s] &= \int\limits_{M^-} \Bigl(\frac{\partial \mathcal{L}^-}{\partial g_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, h_{ij}[i_1 \ldots i_s]\, d^3x,\\ I^+_{g_{ij}}[i_1 \ldots i_s] &= \int\limits_{M^+} \Bigl(\frac{\partial \mathcal{L}^+}{\partial g_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, h_{ij}[i_1 \ldots i_s]\, d^3x, \end{aligned} \tag{6.7}$$

$$\begin{aligned} I^-_{b_{ij}}[i_1 \ldots i_s] &= \int\limits_{M^-} \Bigl(\frac{\partial \mathcal{L}^-}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, \frac{\dot{h}_{ij}[i_1 \ldots i_s]}{2\, c_{\text{gr}}}\, d^3x,\\ I^+_{b_{ij}}[i_1 \ldots i_s] &= \int\limits_{M^+} \Bigl(\frac{\partial \mathcal{L}^+}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, \frac{\dot{h}_{ij}[i_1 \ldots i_s]}{2\, c_{\text{gr}}}\, d^3x. \end{aligned} \tag{6.8}$$

Acting just like as in the previous section and using the formulas (6.7) and (6.8), we derive the following formula for the first order term L_1 in the expansion (6.5):

$$\begin{aligned} L_1 = \sum^{N-1}_{r=0}\,\sum^{N}_{s=r+1}\,\sum^{3}_{i\leqslant j}\,\sum^{3}_{i_1}\ldots\sum^{3}_{i_s}\,\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}} \ldots \partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^+ - \mathcal{L}^-)}{\partial g_{ij}[i_1 \ldots i_s]}\,\cdot\\ \cdot\sqrt{\det g}\,\Bigr)\, h_{ij}[i_1 \ldots i_r]\, \iota_{i_{r+1}}(dx^1 \wedge dx^2 \wedge dx^3)\, - \end{aligned}$$

$$-\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i\leqslant j}^{3}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial}{\partial t}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial b_{ij}[i_1\ldots i_s]}\cdot$$

$$\cdot\sqrt{\det g}\,\Bigr)\,\frac{h_{ij}[i_1\ldots i_r]}{2\,c_{\text{gr}}}\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)\,-$$

$$-\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i\leqslant j}^{3}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial b_{ij}[i_1\ldots i_s]}\cdot$$

$$\cdot\sqrt{\det g}\,\Bigr)\,(\mathbf{u},\mathbf{n})\,\frac{h_{ij}[i_1\ldots i_r]}{2\,c_{\text{gr}}}\,\frac{dS}{\sqrt{\det g}}\,+$$

$$+\sum_{r=1}^{N}\sum_{s=r}^{N}\sum_{i\leqslant j}^{3}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial(\mathcal{L}^{+}-\mathcal{L}^{-})}{\partial b_{ij}[i_1\ldots i_s]}\cdot$$

$$\cdot\sqrt{\det g}\,\Bigr)\,(\mathbf{u},\mathbf{n})\,\frac{h_{ij}[i_1\ldots i_r]}{2\,c_{\text{gr}}}\,\frac{dS}{\sqrt{\det g}}+$$

$$+\sum_{i\leqslant j}^{3}\int\limits_{\sigma}(\mathbf{u},\mathbf{n})\biggl(\Bigl(\frac{\delta\mathcal{L}^{-}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}-\Bigl(\frac{\delta\mathcal{L}^{+}}{\delta\dot{g}_{00}}\Bigr)_{\substack{g,\dot{g},\mathbf{g}\\ \mathbf{Q},\dot{\mathbf{Q}}}}\biggr)\,\frac{h_{ij}}{2\,c_{\text{gr}}}\,dS+\ldots.$$

The above formula[1] is an analog of the formula (5.38). Like in (5.38), the partial differential operators of the form (5.39) should be ignored in those terms of it where $s=r+1$ and $s=r$ respectively. The indices $i_1,\,\ldots\,,\,i_r$ in the above formula, like in (5.38), are subordinate to the indices $i_1,\,\ldots\,,\,i_s$. They obey the inequalities (5.43) produced from (5.9). Using the above formula, we define a series of three-dimensional vector fields $\mathbf{J}_{g_{ij}}[i_1\ldots i_r]$ such that

$$L_1=\sum_{r=0}^{N}\sum_{i\leqslant j}^{3}\sum_{i_1}^{3}\ldots\sum_{i_r}^{3}\mathcal{J}_{g_{ij}}[i_1\ldots i_r]^k\,h_{ij}[i_1\ldots i_r]\,\iota_k(dx^1\wedge dx^2\wedge dx^3)+\ldots. \tag{6.9}$$

The indices $i_1,\,\ldots\,,\,i_r$ in (6.9) also obey the inequalities (5.43). Due to the formula (5.40) the components $\mathcal{J}_{g_{ij}}[i_1\ldots i_r]^k$ of the vector fields $\mathbf{J}_{g_{ij}}[i_1\ldots i_r]$ are uniquely determined by the formula (6.9) and by the formula preceding (6.9). Using these components, we write the boundary conditions

$$\sum_{k=1}^{3}\mathcal{J}_{g_{ij}}[i_1\ldots i_r]^k\,n_k\,\biggr|_{P\in\sigma}=0 \tag{6.10}$$

and then we formulate the following theorem.

Theorem 6.1. *If the stationary action principle is valid for discontinuous Lagrangian densities of the form* (4.2)*, then field configurations of corresponding Lagrangian field theories obey the boundary conditions* (6.10) *on the interface boundary* $\sigma=\partial M^{-}=-\partial M^{+}$.

[1] Note that for $s=0$ in (6.7) we should separately take into account the dependence of $\det g$ on g_{ij}. However the contribution of the arising extra terms goes to ellipses in (6.9) and in the formula preceding the formula (6.9).

7. Boundary conditions associated with matter.

The matter fields in the present paper are given by the functions (3.4) and their time derivatives (3.5). Small variations of the functions (3.4) are written as

$$\hat{Q}_i = Q_i(t, x^1, x^2, x^3) + \varepsilon\, h_i(t, x^1, x^2, x^3). \tag{7.1}$$

Here $\varepsilon \to 0$ is a small parameter, while $h_i(t, x^1, x^2, x^3)$ are arbitrary smooth functions with compact support (see [33]). Differentiating (7.1) with respect to the time variable t, we get the following equality:

$$\dot{\hat{Q}}_i = \dot{Q}_i(t, x^1, x^2, x^3) + \varepsilon\, \dot{h}_i(t, x^1, x^2, x^3). \tag{7.2}$$

Like in the previous section, in this case the formula (5.6) remains unchanged, while the formulas (5.7) and (5.8) in this case are replaced with the formulas

$$Q_i[i_1 \ldots i_s] = \frac{\partial g_{00}}{\partial x^{i_1} \ldots \partial x^{i_s}}, \qquad\qquad h_i[i_1 \ldots i_s] = \frac{\partial h_i}{\partial x^{i_1} \ldots \partial x^{i_s}}, \tag{7.3}$$

$$\dot{Q}_i[i_1 \ldots i_s] = \frac{\partial \dot{g}_{00}}{\partial x^{i_1} \ldots \partial x^{i_s}}, \qquad\qquad \dot{h}_i[i_1 \ldots i_s] = \frac{\partial \dot{h}_i}{\partial x^{i_1} \ldots \partial x^{i_s}}. \tag{7.4}$$

The variations (7.1) and (7.2) lead to the variations

$$\hat{L} = L + \varepsilon\, L_1 + \ldots\,, \qquad\qquad \hat{S} = S + \varepsilon\, S_1 + \ldots \tag{7.5}$$

similar to (5.10), (5.24) and (6.5). By ellipses in (7.5) we denote higher order terms with respect to the small parameter $\varepsilon \to 0$. Due to (4.2) and (5.6) the first order term L_1 in (7.5) is subdivided into two parts

$$L_1 = L_1^- + L_1^+. \tag{7.6}$$

Each derivative $Q_i[i_1 \ldots i_s]$ of the from (7.3) and each derivative $\dot{Q}_i[i_1 \ldots i_s]$ of the form (7.4) makes its own contributions to L_1^- and to L_1^+ in (7.6). Let's denote these contributions through $I^-_{Q_i}[i_1 \ldots i_s]$, $I^+_{Q_i}[i_1 \ldots i_s]$, $I^-_{\dot{Q}_i}[i_1 \ldots i_s]$, and $I^+_{\dot{Q}_i}[i_1 \ldots i_s]$. In explicit form these contributions are given by the following integrals:

$$\begin{aligned} I^-_{Q_i}[i_1 \ldots i_s] &= \int\limits_{M^-} \Bigl(\frac{\partial \mathcal{L}^-}{\partial Q_i[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, h_i[i_1 \ldots i_s]\, d^3x,\\ I^+_{Q_i}[i_1 \ldots i_s] &= \int\limits_{M^+} \Bigl(\frac{\partial \mathcal{L}^+}{\partial Q_i[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, h_i[i_1 \ldots i_s]\, d^3x, \end{aligned} \tag{7.7}$$

$$\begin{aligned} I^-_{\dot{Q}_i}[i_1 \ldots i_s] &= \int\limits_{M^-} \Bigl(\frac{\partial \mathcal{L}^-}{\partial \dot{Q}_i[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, \dot{h}_i[i_1 \ldots i_s]\, d^3x,\\ I^+_{\dot{Q}_i}[i_1 \ldots i_s] &= \int\limits_{M^+} \Bigl(\frac{\partial \mathcal{L}^+}{\partial \dot{Q}_i[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\, \dot{h}_i[i_1 \ldots i_s]\, d^3x. \end{aligned} \tag{7.8}$$

Acting just like as in two previous sections and using the formulas (7.7) and (7.8), we derive the following formula for the first order term L_1 in the expansion (7.5):

$$
\begin{gathered}
L_1=\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i=1}^{n}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial Q_i[i_1\ldots i_s]}\,\cdot\\
\cdot\sqrt{\det g}\,\biggr)\,h_i[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)\,-\\
-\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i=1}^{n}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial}{\partial t}\,\frac{\partial^{\,s-r-1}}{\partial x^{i_{r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial \dot Q_i[i_1\ldots i_s]}\,\cdot\\
\cdot\sqrt{\det g}\,\biggr)\,h_i[i_1\ldots i_r]\,\iota_{i_{r+1}}(dx^1\wedge dx^2\wedge dx^3)\,-\\
-\sum_{r=0}^{N-1}\sum_{s=r+1}^{N}\sum_{i=1}^{n}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\biggl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial \dot Q_i[i_1\ldots i_s]}\,\cdot\\
\cdot\sqrt{\det g}\,\biggr)\,(\mathbf{u},\mathbf{n})\,h_i[i_1\ldots i_r]\,\frac{dS}{\sqrt{\det g}}\,+\\
+\sum_{r=1}^{N}\sum_{s=r}^{N}\sum_{i=1}^{n}\sum_{i_1}^{3}\ldots\sum_{i_s}^{3}\int\limits_{\sigma}(-1)^{s-r}\frac{\partial^{\,s-r}}{\partial x^{i_{r+1}}\ldots\partial x^{i_s}}\biggl(\frac{\partial(\mathcal{L}^+-\mathcal{L}^-)}{\partial \dot Q_i[i_1\ldots i_s]}\,\cdot\\
\cdot\sqrt{\det g}\,\biggr)\,(\mathbf{u},\mathbf{n})\,h_i[i_1\ldots i_r]\,\frac{dS}{\sqrt{\det g}}\,+\\
+\sum_{i=1}^{n}\int\limits_{\sigma}(\mathbf{u},\mathbf{n})\biggl(\Bigl(\frac{\delta\mathcal{L}^-}{\delta\dot Q_i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}-\Bigl(\frac{\delta\mathcal{L}^+}{\delta\dot Q_i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},\mathbf{Q}}}\biggr)\,h_i\,dS+\ldots.
\end{gathered}
$$

The above formula is an analog of the formula (5.38). Like in (5.38), the partial differential operators of the form (5.39) should be ignored in those terms of it where $s=r+1$ and $s=r$ respectively. The indices $i_1,\,\ldots\,,\,i_r$ in the above formula, like in (5.38), are subordinate to the indices $i_1,\,\ldots\,,\,i_s$. They obey the inequalities (5.43) produced from (5.9). Using the above formula, we define a series of three-dimensional vector fields $\mathbf{J}_{Q_i}[i_1\ldots i_r]$ such that

$$
L_1=\sum_{r=0}^{N}\sum_{i=1}^{n}\sum_{i_1}^{3}\ldots\sum_{i_r}^{3}\mathcal{J}_{Q_i}[i_1\ldots i_r]^k\,h_i[i_1\ldots i_r]\,\iota_k(dx^1\wedge dx^2\wedge dx^3)+\ldots. \tag{7.9}
$$

Like in section 5, the vector fields $\mathbf{J}_{Q_i}[i_1\ldots i_r]$ here are defined not globally in the universe M, but only at the points of the surface σ, though they are not inner vector fields of this surface.

The indices $i_1,\,\ldots\,,\,i_r$ in (7.9) obey the inequalities (5.43). Due to the formula (5.40) the components $\mathcal{J}_{Q_i}[i_1\ldots i_r]^k$ of the vector fields $\mathbf{J}_{Q_i}[i_1\ldots i_r]$ are uniquely determined by the formula (7.9) and by the formula preceding (7.9). Using these components, we write the boundary conditions

$$
\sum_{k=1}^{3}\mathcal{J}_{Q_i}[i_1\ldots i_r]^k\,n_k\,\biggr|_{P\in\sigma}=0 \tag{7.10}
$$

and then we formulate the following theorem.

Theorem 7.1. *If the stationary action principle is valid for discontinuous Lagrangian densities of the form* (4.2)*, then field configurations of corresponding Lagrangian field theories obey the boundary conditions* (7.10) *on the interface boundary* $\sigma = \partial M^{-} = -\partial M^{+}$.

8. Concluding remarks.

Theorems 5.1, 6.1, and 7.1 constitute the main result of the present paper. These theorems contain the condition «if the stationary action principle is valid for discontinuous Lagrangian densities» in their statements. In support of this «if» one can mention the papers [34–36] on Lagrangian field theories with boundaries.

9. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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