RELATIVISTIC ELASTICITY IN THE 3D-BRANE UNIVERSE MODEL. PART 2.

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Abstract. The 3D-brane universe model is the name of a new non-Einsteinian theory of gravity. Recently, nonlinear relativistic elastic media were considered within this theory. In this paper, we consider such media separated by an interface and derive boundary conditions for them.

1. Introduction.

The 3D-brane universe model is a new non-Einsteinian theory of gravity assuming the real physical universe M to be a three-dimensional manifold equipped with a time-dependent Riemannian metric

$$g_{ij}=g_{ij}(t,x^1,x^2,x^3),\quad 1\leqslant i,j\leqslant 3. \tag{1.1}$$

Unlike Einstein's theory, in the new theory the four-dimensional spacetime M_4 is not a physical continuum. It is just a mathematical abstraction that has no physical existence. This non-physical spacetime serves as a bridge connecting the new theory with Einstein's relativity. It holds four geometric structures:

1) a pseudo-Riemannian metric with the signature $(+,-,-,-)$;
2) an orientation;
3) a polarization;
4) a foliation of spacelike 3D-branes filling the spacetime entirely with the exception of perhaps one point corresponding to the Big Bang;

three of which are inherited from Einstein's theory and the fourth is a complementary one introduced in the new theory (see § 3 of Chapter III in the textbook [1] and § 2 of Chapter I in the monograph [2] or in its English translation [3]). More references on the new theory are e-prints [4–18] and conference abstracts [19–36].

Due to the above four geometric structures in the spacetime M_4, there is a class of local coordinates in the real universe M, which are called comoving coordinates, and there is a class of global time variables, each of which is called membrane time (or brane time for short) (see § 3 and § 5 of Chapter I in [3]). The coordinates x^1, x^2, x^3 and the time variable t in (1.1) are representatives of these two classes. We shall use these coordinates and this time variable of the universe M throughout the present paper.

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In [17], within the Lagrangian approach and using comoving coordinates x^1, x^2, x^3 and brane time t, the relativistic dynamics of elastic media in their bulk was described. In the present paper, we continue this research and derive boundary conditions at an interface of two elastic media and/or at an interface of an elastic medium and vacuum. A general approach to deriving boundary conditions at interfaces of two arbitrary media was developed in [18].

2. Dynamical variables of the gravitational field.

Upon choosing some comoving coordinates x^1, x^2, x^3 and some brane time variable t in M, we produce four-dimensional coordinates in the spacetime M_4:

$$x^0 = c_{\text{gr}}\, t, \qquad x^1, \qquad x^2, \qquad x^3. \tag{2.1}$$

Here c_{gr} is a speed constant[1] analogous to the speed of light (see § 1 of Chapter II in [3]). In the coordinates (2.1), the pseudo-Riemannian metric of the spacetime M_4 is given by the block-diagonal matrix

$$\begin{Vmatrix} g_{00} & 0 & 0 & 0\\ 0 & -g_{11} & -g_{12} & -g_{13}\\ 0 & -g_{21} & -g_{22} & -g_{23}\\ 0 & -g_{31} & -g_{32} & -g_{33}\end{Vmatrix}. \tag{2.2}$$

The upper-left diagonal element of the matrix (2.2) is interpreted as a scalar function in the real physical universe M:

$$g_{00}(t, x^1, x^2, x^3). \tag{2.3}$$

Other components of the matrix (2.2) with negative signs are taken from (1.1). The functions (1.1) and (2.3) constitute the set of dynamical variables of the gravitational field in the new theory.

3. Dynamical variables of an elastic medium.

As in [17], here, we assume that some elastic solid medium was melted and then frozen again in a three-dimensional space with the coordinates y^1, y^2, y^3 and with the three-dimensional metric

$$\eta_{ij} = \eta_{ij}(y^1, y^2, y^3), \quad 1 \leqslant i, j \leqslant 3. \tag{3.1}$$

Then we assume that this medium is immersed in the real universe M, which is described by comoving coordinates x^1, x^2, x^3, by brane time t, by scalar function (2.3), and by metric (1.1). The immersion is described by the functions

$$\begin{cases} x^1 = x^1(t, y^1, y^2, y^3),\\ x^2 = x^2(t, y^1, y^2, y^3),\\ x^3 = x^3(t, y^1, y^2, y^3). \end{cases} \tag{3.2}$$

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[1] In the new theory, the speed constant c_{gr} does not necessarily coincide with the speed of light c_{el}, i. e., with the speed of electromagnetic waves.

As in [17], the functions (3.2) are assumed to produce an invertible mapping whose inverse mapping is given by three similar functions

$$\begin{cases} y^1 = y^1(t, x^1, x^2, x^3), \\ y^2 = y^2(t, x^1, x^2, x^3), \\ y^3 = y^3(t, x^1, x^2, x^3). \end{cases} \tag{3.3}$$

In [17], the functions (3.3) were chosen as the dynamical variables of the elastic medium. In the context of the paper [18], they replace the functions $Q_1, \ldots, Q_n$, i. e., here we have $n = 3$ and $Q_1 = y^1$, $Q_2 = y^2$, $Q_3 = y^3$.

4. Action integrals and Lagrangians.

Action integrals in field theories are written as time integrals of Lagrangians, while Lagrangians are spatial integrals of Lagrangian densities:

$$S = \int L\,dt, \qquad\qquad L = \int \mathcal{L}\,\sqrt{\det g}\;d^3x. \tag{4.1}$$

In relativistic elasticity, which is developed within the framework of the 3D-brane universe model, the Lagrangian density $\mathcal{L}$ is the sum of two terms:

$$\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}}. \tag{4.2}$$

The same decomposition applies to the Lagrangian L and to the action integral S:

$$L = L_{\text{gr}} + L_{\text{mat}}, \qquad\qquad S = S_{\text{gr}} + S_{\text{mat}}, \tag{4.3}$$

where

$$L_{\text{gr}} = \int \mathcal{L}_{\text{gr}}\,\sqrt{\det g}\;d^3x, \qquad\qquad L_{\text{mat}} = \int \mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x, \tag{4.4}$$

$$S_{\text{gr}} = \int L_{\text{gr}}\,dt, \qquad\qquad S_{\text{mat}} = \int L_{\text{mat}}\,dt. \tag{4.5}$$

The formulas (4.3) are easily derived from (4.2) using (4.1), (4.4), and (4.5). The integrals on the left in (4.4) and (4.5) are associated with the gravitational field, while the integrals on the right are associated with matter. In this paper, matter is represented by elastic media.

The Lagrangian density $\mathcal{L}_{\text{gr}}$ of the gravitational field is given by the formula

$$\mathcal{L}_{\text{gr}} = -\frac{c_{\text{gr}}^4}{16\,\pi\,\gamma}\,\sqrt{g_{00}}\,(\mathfrak{r} + 2\,\Lambda), \tag{4.6}$$

where

$$\begin{aligned} \mathfrak{r} = g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_{kq}\,g_{00} - \frac{g_{00}^{-2}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3} g^{kq}\,\nabla_k\,g_{00}\,\nabla_q\,g_{00} \,- \\ - R - g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\,b^q_k + g_{00}^{-1}\sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\,b^q_q \end{aligned} \tag{4.7}$$

(see [12] and § 3 of Chapter III in [3]). In these sources, the symbol ρ was used instead of $\mathfrak{r}$, but here we reserve the symbol ρ for a different purpose.

Here, γ in (4.6) denotes Newton's gravitational constant (see [37]), while Λ is the cosmological constant (see [38]). Note that our choice for the sign of Λ is opposite to that used in [38]; this convention is adopted to ensure consistency with the notations of [1–3].

Here, ∇ in (4.7) denotes the covariant differentiation with respect to the Levi-Civita metric connection Γ^k_{ij} determined by the three-dimensional metric (1.1):

$$\Gamma^k_{ij}=\frac{1}{2}\sum_{q=1}^{3}g^{kq}\left(\frac{\partial g_{qj}}{\partial x^i}+\frac{\partial g_{iq}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^q}\right) \tag{4.8}$$

(see [39], or § 7 of Chapter III in [40]). The symbol R denotes the three-dimensional scalar curvature of the connection (4.8) (see [41], or § 8 of Chapter IV in [40]).

The components of the tensor field $\mathbf{b}$ in (4.7) are given by the formulas

$$b_{ij}=\frac{\dot g_{ij}}{2\,c_{\text{gr}}}=\frac{1}{2\,c_{\text{gr}}}\,\frac{\partial g_{ij}}{\partial t},\qquad\qquad b^k_q=\sum_{s=1}^{3}g^{ks}\,b_{sq}. \tag{4.9}$$

The Lagrangian density $\mathcal{L}_{\text{mat}}$ of an elastic medium is given by the formula

$$\mathcal{L}_{\text{mat}}=-\rho\,c_{\text{br}}^2\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}-\mathcal{F}(y,J,\mathbf{v},\mathbf{u}). \tag{4.10}$$

Here, c_{br} is a speed constant analogous to c_{gr} in (4.9) and (2.1). It is interpreted as the critical speed of baryonic matter[1] that constitutes the elastic medium. As with c_{gr}, in the present theory, the constant c_{br} does not necessarily coincide with the speed of light c_{el}. If the elastic medium consists of non-baryonic dark matter, the constant c_{br} in (4.10) should be replaced by c_{nb}.

Apart from c_{br}, (4.10) contains several other notations introduced in [17]. To recall these notations, we rewrite (3.2) and (3.3) as

$$\begin{cases}t=\tau,\\ x^1=x^1(\tau,y^1,y^2,y^3),\\ x^2=x^2(\tau,y^1,y^2,y^3),\\ x^3=x^3(\tau,y^1,y^2,y^3),\end{cases}\qquad\qquad \begin{cases}\tau=t,\\ y^1=y^1(t,x^1,x^2,x^3),\\ y^2=y^2(t,x^1,x^2,x^3),\\ y^3=y^3(t,x^1,x^2,x^3).\end{cases} \tag{4.11}$$

The components of the velocity vector $\mathbf{v}$ in (4.10) are defined as the derivatives

$$v^i=\dot x^i(\tau,y^1,y^2,y^3)=\frac{\partial x^i(\tau,y^1,y^2,y^3)}{\partial\tau},\quad 1\leqslant i\leqslant 3. \tag{4.12}$$

The components of the velocity vector $\mathbf{v}$ in (4.12) are functions of the variables τ, y^1, y^2, y^3. However, to treat $\mathbf{v}$ as a vector field in the physical universe, we

[1] In astrophysics, the term «baryonic matter» refers to ordinary matter, excluding dark matter.

express (4.12) in terms of the coordinates t, x^1, x^2, x^3 by using the second set of relations in (4.11). Consequently, in (4.10), they appear as the functions

$$v^i = v^i(t, x^1, x^2, x^3), \quad 1 \leqslant i \leqslant 3. \tag{4.13}$$

The vector field $\mathbf{v}$ with components (4.13) describes the physical motion of the elastic medium in the three-dimensional universe.

In [17], two Jacobian matrices were defined by using the sets of functions (4.11):

$$M^r_m = \frac{\partial x^r}{\partial y^m}, \qquad\qquad J^i_j = \frac{\partial y^i}{\partial x^j}. \tag{4.14}$$

The matrices in (4.14) are mutually inverse. As noted above, our elastic medium is assumed to have been melted and subsequently frozen in a static three-dimensional space with the metric (3.1). Since this medium is at rest in that space, its rest mass density is given by a function that does not depend on τ:

$$\rho_0(y^1, y^2, y^3). \tag{4.15}$$

By using the second set of functions in (4.11), we can pull back the metric (3.1) from the static three-dimensional space to the physical three-dimensional universe:

$$G_{ij} = \sum_{r=1}^{3}\sum_{s=1}^{3} \eta_{rs}\,\frac{\partial y^r}{\partial x^i}\,\frac{\partial y^s}{\partial x^j}, \quad 1 \leqslant i, j \leqslant 3. \tag{4.16}$$

According to [17] (see formula (5.6) therein), the rest mass density ρ of our elastic medium is given by the formula

$$\rho = \rho(t, x^1, x^2, x^3) = \rho_0\,\frac{\sqrt{\det G}}{\sqrt{\det g}}\,, \tag{4.17}$$

where ρ_0 is introduced in (4.15). This density enters the Lagrangian density (4.10).

By using (4.17), the following differential equation for the rest mass density ρ was derived in [17]:

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{3} c_{\mathrm{gr}}\,\rho\,b^k_k + \sum_{i=1}^{3} \nabla_i(\rho\,v^i) = 0. \tag{4.18}$$

Equation (4.18) is purely kinematic; the Lagrangian density (4.10) is not used in its derivation.

Apart from (4.18), another equation was derived in [17], which reads as

$$\frac{\partial p_n}{\partial t} + \sum_{k=1}^{3} c_{\mathrm{gr}}\,p_n\,b^k_k + \sum_{m=1}^{3} \nabla_m \Pi^m_n = f_n + \mathcal{F}_n. \tag{4.19}$$

Here,

$$\mathcal{F}_n = \sum_{i=1}^{3}\biggl(-\frac{\partial}{\partial t}\Bigl(\frac{\delta\mathcal{F}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}} - c_{\mathrm{gr}}\,\Bigl(\frac{\delta\mathcal{F}}{\delta w^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},y}} \sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{F}}{\delta y^i}\Bigr)_{\substack{g,\dot g,\mathbf{g}\\ \mathbf{b},w}}\biggr)\,J^i_n. \tag{4.20}$$

The quantities f_n on the right-hand side of (4.19) are given by the formula

$$f_n = -\frac{1}{2}\,\frac{\rho\,c_{\text{br}}^2\,\nabla_n g_{00}}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,. \tag{4.21}$$

The symbols p_n in (4.19) denote the components of the momentum density covector of the elastic medium, while Π^m_n are the components of the momentum flux density tensor. They are given by the formulas

$$p_n = \frac{\rho\,v_n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,, \qquad\qquad \Pi^m_n = \frac{\rho\,v_n\,v^m}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}\,.$$

The quantities $\mathcal{F}_n$ and f_n in (4.20) and (4.21) are interpreted as the components of force density covectors. Specifically, the quantities f_n are associated with gravitational forces, whereas $\mathcal{F}_n$ are associated with elastic response forces.

Unlike (4.18), Equation (4.19) is a dynamical equation. It is derived by applying the principle of stationary action (see [42]) to the action integral S in (4.1) with respect to the dynamical variables y^1, y^2, y^3 from (3.3) and (4.11), as well as their time derivatives $w^1 = \dot y^1$, $w^2 = \dot y^2$, and $w^3 = \dot y^3$.

5. Interface of two elastic media.

As in [18], we consider two domains M^+ and M^- in the physical three-dimensional universe M separated by a two-dimensional moving boundary σ (see Fig. 5.1). The domains are assumed to be filled with two distinct elastic media, the boundary σ serving as their interface. The Lagrangian density for such a pair of media is

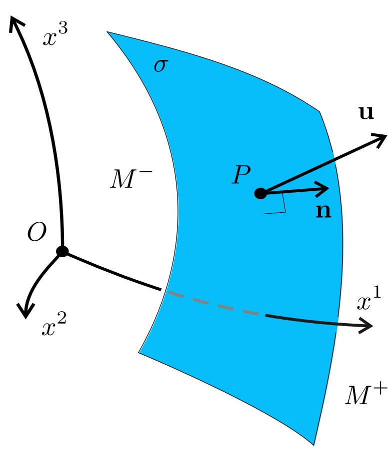

Fig. 5.1

$$\mathcal{L} = \begin{cases} \mathcal{L}^- & \text{if } P \in M^-, \\ \mathcal{L}^+ & \text{if } P \in M^+. \end{cases} \tag{5.1}$$

The Lagrangian densities $\mathcal{L}^-$ and $\mathcal{L}^+$ in (5.1) share the same gravitational part $\mathcal{L}_{\text{gr}}$ introduced in (4.2), i. e.,

$$\begin{aligned} \mathcal{L}^- &= \mathcal{L}_{\text{gr}} + \mathcal{L}^-_{\text{mat}}, \\ \mathcal{L}^+ &= \mathcal{L}_{\text{gr}} + \mathcal{L}^+_{\text{mat}}. \end{aligned} \tag{5.2}$$

The term $\mathcal{L}_{\text{gr}}$ in both relations in (5.2) is given by (4.6), whereas $\mathcal{L}^-_{\text{mat}}$ is defined by

$$\mathcal{L}^-_{\text{mat}} = -\rho^-\,c_{\text{br}}^2\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}} - \mathcal{F}^-(y, J, \mathbf{v}, \mathbf{u}) \tag{5.3}$$

analogously to (4.10). The term $\mathcal{L}^+_{\text{mat}}$ in the second relation in (5.2) is likewise given by an analogous expression:

$$\mathcal{L}^{+}_{\text{mat}} = -\rho^{+}\, c^{2}_{\text{br}} \sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c^{2}_{\text{br}}}} - \mathcal{F}^{+}(y, J, \mathbf{v}, \mathbf{u}). \tag{5.4}$$

Note that the symbol $\mathbf{u}$ has different meanings in (4.10), (5.3), (5.4), and Fig. 5.1. In Fig. 5.1, $\mathbf{u}$ stands for the velocity vector of an arbitrary point $P \in \sigma$, whereas in (4.10), (5.3), and (5.4), it denotes the deformation tensor, which will be considered below. The use of the same notation for two distinct concepts is due to a limited choice of symbols traditionally adopted for these physical quantities.

6. Small variations of the dynamical variables of matter.

Above, we have chosen y^1, y^2, y^3 from (3.3) and (4.11) as the dynamical variables of the elastic medium. Their time derivatives w^1, w^2, w^3 are given by

$$w^i = \dot{y}^i = \frac{\partial y^i(t, x^1, x^2, x^3)}{\partial t}, \quad 1 \leqslant i \leqslant 3. \tag{6.1}$$

The variables (6.1) were already used in (4.20). Now we consider small variations of the functions y^1, y^2, y^3, which are given by

$$\hat{y}^i = y^i(t, x^1, x^2, x^3) + \varepsilon\, h^i(t, x^1, x^2, x^3), \tag{6.2}$$

where $\varepsilon \to 0$ and $h^i(t, x^1, x^2, x^3)$ are arbitrary smooth functions with compact support with respect to all their arguments (see [43]). From (6.2), for the time derivatives (6.1) we obtain

$$\hat{w}^i = w^i(t, x^1, x^2, x^3) + \varepsilon\, \dot{h}^i(t, x^1, x^2, x^3). \tag{6.3}$$

Note that the time derivatives (6.1) are not the components of the velocity vector $\mathbf{v}$, which are given by (4.13) and obtained from (4.12) by applying (4.11). In [17], the following relation was derived (see formula (7.3) therein) to express v^1, v^2, v^3 in terms of w^1, w^2, w^3:

$$v^i = -\sum_{r=1}^{3} M^i_r\, w^r. \tag{6.4}$$

Here, M^i_r are the components of the first Jacobian matrix in (4.14). To apply (6.2) and (6.4), we recall formula (7.11) from [17]:

$$\hat{M}^i_r = M^i_r - \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} M^i_m\, \frac{\partial h^m}{\partial x^n}\, M^n_r + \ldots\,. \tag{6.5}$$

Since the indices m and r in (6.5) are not tensorial indices associated with the coordinates x^1, x^2, x^3 in the physical universe, we can rewrite (6.5) by using the covariant derivative:

$$\hat{M}^i_r = M^i_r - \varepsilon \sum_{m=1}^{3} \sum_{n=1}^{3} M^i_m\, \nabla_{\!n} h^m\, M^n_r + \ldots\,. \tag{6.6}$$

The ellipses in (6.5) and (6.6) denote higher-order terms with respect to the small parameter $\varepsilon \to 0$. By applying (6.3) and (6.6) to (6.4), we obtain

$$\hat{v}^i = v^i - \varepsilon \sum_{r=1}^{3} M^i_r\,\dot{h}^r - \varepsilon \sum_{m=1}^{3}\sum_{n=1}^{3} M^i_m\,v^n\,\nabla_n h^m + \ldots . \tag{6.7}$$

Relation (6.7) differs slightly from formula (7.14) in [17]. This discrepancy arises because, in [17], y^1, y^2, y^3 and w^1, w^2, w^3 were treated as dynamical variables with independent variations.

Note that the Lagrangian density $\mathcal{L}_{\text{mat}}$ in (4.10) is composed of two parts. Following [17], we denote its first part by $\mathcal{K}$:

$$\mathcal{K} = -\rho\,c_{\text{br}}^2 \sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}, \qquad\qquad \mathcal{L}_{\text{mat}} = \mathcal{K} - \mathcal{F}. \tag{6.8}$$

The term $\mathcal{K}$ in (6.8) is interpreted as the kinetic energy density, while $\mathcal{F}$ is interpreted as the potential energy density.

Now consider the density ρ in (6.8), which is given by (4.17). The right-hand side of (4.17) does not depend on the time derivatives (6.1). Therefore, applying (6.2) to (4.17) and repeating the arguments from [17], we obtain

$$\hat{\rho} = \rho\left(1 + \varepsilon \sum_{m=1}^{3}\sum_{n=1}^{3} M^n_m\,\frac{\nabla_n(h^m\,\rho\sqrt{\det g}/|\det J|)}{\rho\sqrt{\det g}/|\det J|}\right) + \ldots . \tag{6.9}$$

Here, J is the Jacobian matrix with components J^i_j defined in (4.14). Relation (6.9) coincides with formula (7.21) in [17].

Apart from ρ, the expression for $\mathcal{K}$ in (6.8) contains a square-root factor. By applying (6.2) and (6.3) to this factor and taking into account (6.7), we obtain

$$\begin{gathered} c_{\text{br}}^2 \sqrt{g_{00} - \frac{|\hat{\mathbf{v}}|^2}{c_{\text{br}}^2}} = c_{\text{br}}^2 \sqrt{g_{00} - \frac{|\mathbf{v}|^2}{c_{\text{br}}^2}} + \\ + \varepsilon \sum_{i=1}^{3}\sum_{r=1}^{3} \frac{v_i\,M^i_r\,\dot{h}^r}{\sqrt{g_{00} - \dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}} + \varepsilon \sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} \frac{v_i\,M^i_m\,\nabla_n h^m\,v^n}{\sqrt{g_{00} - \dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}} + \ldots . \end{gathered} \tag{6.10}$$

Although relation (6.10) is closely related to formula (7.22) in [17], it differs slightly from it.

7. Relativistic deformation tensor.

As noted above, the symbol $\mathbf{u}$ in (4.10), (5.3), and (5.4) denotes the deformation[1] tensor. However, $\mathbf{u}$ is not a standard deformation tensor; first, it is nonlinear, and second, it is relativistic.

To introduce the nonlinear relativistic deformation tensor, following [17], we define two operators: the relativistic contraction operator C and the relativistic

[1] In literature of Russian origin, the term «deformation tensor» is commonly used as a synonym for «strain tensor». In [17] and the present paper, we follow this convention.

extension operator E. These operators are mutually inverse and are given by two square matrices with the components

$$C^i_j=\delta^i_j-\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(1-\sqrt{1-\frac{|\mathbf{v}|^2}{g_{00}\,c_{\mathrm{br}}^2}}\,\right),\tag{7.1}$$

$$E^i_j=\delta^i_j-\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(1-\frac{1}{\sqrt{1-\frac{|\mathbf{v}|^2}{g_{00}\,c_{\mathrm{br}}^2}}}\right)\tag{7.2}$$

(see [17] and the erratum thereto). Here, δ^i_j denotes the Kronecker delta (see [44]). By applying the operator (7.2) to the metric (1.1), we obtain a new metric:

$$\breve{g}_{ij}=\sum_{r=1}^{3}\sum_{s=1}^{3}E^r_i\,g_{rs}\,E^s_j,\quad 1\leqslant i,j\leqslant 3.\tag{7.3}$$

The metric (7.3) defines the nonlinear relativistic deformation tensor

$$u_{ij}=\frac{\breve{g}_{ij}-G_{ij}}{2}.\tag{7.4}$$

Here, G_{ij} are the components of the metric defined in (4.16). Relation (7.4) coincides with formula (3.11) in [17] and is similar to formula (4.11) in [45].

The main objective of this section is to apply the small variations (6.2) and (6.3) to the deformation tensor (7.4). This will be accomplished in several steps.

From (6.7), we obtain

$$\begin{gathered}\hat{v}^i\,\hat{v}_j=v^i\,v_j-\varepsilon\sum_{r=1}^{3}M^i_r\,v_j\,\dot{h}^r-\varepsilon\sum_{m=1}^{3}\sum_{n=1}^{3}M^i_m\,v^n\,v_j\,\nabla_n h^m-\\-\varepsilon\sum_{s=1}^{3}\sum_{r=1}^{3}v^i\,M^s_r\,g_{sj}\,\dot{h}^r-\varepsilon\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}v^i\,M^s_m\,g_{sj}\,v^n\,\nabla_n h^m+\ldots.\end{gathered}\tag{7.5}$$

By applying tensor contraction (see [46]) to (7.5), we obtain

$$|\hat{\mathbf{v}}|^2=|\mathbf{v}|^2-\varepsilon\sum_{i=1}^{3}\sum_{r=1}^{3}2\,M^i_r\,v_i\,\dot{h}^r-\varepsilon\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}2\,M^i_m\,v^n\,v_i\,\nabla_n h^m+\ldots.\tag{7.6}$$

The next step is to apply (7.6) to the square-root factors in (7.1) and (7.2):

$$\begin{gathered}\sqrt{1-\frac{|\hat{\mathbf{v}}|^2}{g_{00}\,c_{\mathrm{br}}^2}}=\sqrt{1-\frac{|\mathbf{v}|^2}{g_{00}\,c_{\mathrm{br}}^2}}+\frac{1}{\sqrt{1-\frac{|\mathbf{v}|^2}{g_{00}\,c_{\mathrm{br}}^2}}}\cdot\frac{\varepsilon}{g_{00}\,c_{\mathrm{br}}^2}\cdot\\\cdot\left(\sum_{i=1}^{3}\sum_{r=1}^{3}M^i_r\,v_i\,\dot{h}^r+\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}M^i_m\,v^n\,v_i\,\nabla_n h^m\right)+\ldots.\end{gathered}\tag{7.7}$$

Relation (7.2) contains the reciprocal of the square root in (7.7). By applying (7.6), we obtain the variation of this reciprocal factor:

$$\begin{gathered}\frac{1}{\sqrt{1-\dfrac{|\hat{\mathbf{v}}|^2}{g_{00}\,c_{\text{br}}^2}}}=\frac{1}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}}-\frac{1}{\left(\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}\right)^3}\cdot\frac{\varepsilon}{g_{00}\,c_{\text{br}}^2}\,\cdot\\ \cdot\left(\sum_{i=1}^{3}\sum_{r=1}^{3}M^i_r\,v_i\,\dot{h}^r+\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}M^i_m\,v^n\,v_i\,\nabla_n h^m\right)+\ldots.\end{gathered}\tag{7.8}$$

Relations (7.7) and (7.8) are analogous. In addition to these expansions, we require the following relation:

$$\frac{1}{|\hat{\mathbf{v}}|^2}=\frac{1}{|\mathbf{v}|^2}+\varepsilon\sum_{i=1}^{3}\sum_{r=1}^{3}\frac{2\,M^i_r\,v_i\,\dot{h}^r}{|\mathbf{v}|^4}+\varepsilon\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{2\,M^i_m\,v^n\,v_i\,\nabla_n h^m}{|\mathbf{v}|^4}+\ldots.\tag{7.9}$$

Now we apply (7.5), (7.7), and (7.9) to calculate the variation of the relativistic contraction operator (7.1):

$$\begin{gathered}\hat{C}^i_j=C^i_j-\varepsilon\,\frac{v^i\,v_j}{|\mathbf{v}|^4}\left(2-\sqrt{1-\frac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}-\frac{1}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}}\right)\cdot\\ \cdot\left(\sum_{s=1}^{3}\sum_{r=1}^{3}M^s_r\,v_s\,\dot{h}^r+\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}M^s_m\,v^n\,v_s\,\nabla_n h^m\right)+\\ +\frac{\varepsilon}{|\mathbf{v}|^2}\left(1-\sqrt{1-\frac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}\right)\left(\sum_{r=1}^{3}M^i_r\,v_j\,\dot{h}^r+\sum_{m=1}^{3}\sum_{n=1}^{3}M^i_m\,v^n\,v_j\,\nabla_n h^m+\right.\\ \left.+\sum_{s=1}^{3}\sum_{r=1}^{3}v^i\,M^s_r\,g_{sj}\,\dot{h}^r+\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}v^i\,M^s_m\,g_{sj}\,v^n\,\nabla_n h^m\right)+\ldots.\end{gathered}\tag{7.10}$$

Similarly, we apply (7.5), (7.8), and (7.9) to calculate the variation of the relativistic extension operator (7.2):

$$\begin{gathered}\hat{E}^i_j=E^i_j-\varepsilon\,\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(2-\frac{3}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}}+\frac{1}{\left(\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}\right)^3}\right)\cdot\\ \cdot\left(\sum_{s=1}^{3}\sum_{r=1}^{3}M^s_r\,v_s\,\dot{h}^r+\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}M^s_m\,v^n\,v_s\,\nabla_n h^m\right)+\\ +\frac{\varepsilon}{|\mathbf{v}|^2}\left(1-\frac{1}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}}\right)\cdot\left(\sum_{r=1}^{3}M^i_r\,v_j\,\dot{h}^r+\sum_{m=1}^{3}\sum_{n=1}^{3}M^i_m\,v^n\,v_j\,\nabla_n h^m+\right.\\ \left.+\sum_{s=1}^{3}\sum_{r=1}^{3}v^i\,M^s_r\,g_{sj}\,\dot{h}^r+\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}v^i\,M^s_m\,g_{sj}\,v^n\,\nabla_n h^m\right)+\ldots.\end{gathered}\tag{7.11}$$

Relations (7.10) and (7.11) are rather cumbersome. The functions h^i from (6.2) enter these relations via the time derivatives $\dot{h}^i$ and the covariant derivatives $\nabla_n h^i$. Since y^1, y^2, y^3 in (6.2) are coordinates in the static space rather than the physical universe, the upper index i in h^i is not a tensorial index. Consequently,

$$\nabla_n h^i = \frac{\partial h^i}{\partial x^n}. \tag{7.12}$$

Taking into account (7.12), we rewrite (7.11) as

$$\hat{E}^i_j = E^i_j + \varepsilon \sum_{m=1}^{3} \frac{\partial E^i_j}{\partial w^m}\,\dot{h}^m + \varepsilon \sum_{m=1}^{3}\sum_{n=1}^{3} \frac{\partial E^i_j}{\partial J^m_n}\,\nabla_n h^m + \ldots . \tag{7.13}$$

The partial derivatives in (7.13) serve as shorthand notations for the following expressions:

$$\begin{gathered}\frac{\partial E^i_j}{\partial w^m} = -\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(2 - \frac{3}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}} + \frac{1}{\left(\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}\right)^3}\right)\cdot\sum_{s=1}^{3} M^s_m\,v_s + {}\\ +\frac{1}{|\mathbf{v}|^2}\left(1 - \frac{1}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}}\right)\cdot\left(M^i_m\,v_j + \sum_{s=1}^{3} v^i\,M^s_m\,g_{sj}\right),\end{gathered} \tag{7.14}$$

$$\begin{gathered}\frac{\partial E^i_j}{\partial J^m_n} = -\frac{v^i\,v_j}{|\mathbf{v}|^2}\left(2 - \frac{3}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}} + \frac{1}{\left(\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}\right)^3}\right)\cdot\sum_{s=1}^{3} M^s_m\,v^n\,v_s + {}\\ +\frac{1}{|\mathbf{v}|^2}\left(1 - \frac{1}{\sqrt{1-\dfrac{|\mathbf{v}|^2}{g_{00}\,c_{\text{br}}^2}}}\right)\cdot\left(M^i_m\,v^n\,v_j + \sum_{s=1}^{3} v^i\,M^s_m\,g_{sj}\,v^n\right).\end{gathered} \tag{7.15}$$

Relations (7.14) and (7.15) are derived from (7.11) by using (7.13).

Now we consider the metric (7.3) and apply (7.13) to it:

$$\hat{\breve{g}}_{ij} = \breve{g}_{ij} + \varepsilon \sum_{m=1}^{3} \frac{\partial \breve{g}_{ij}}{\partial w^m}\,\dot{h}^m + \varepsilon \sum_{m=1}^{3}\sum_{n=1}^{3} \frac{\partial g_{ij}}{\partial J^m_n}\,\nabla_n h^m + \ldots . \tag{7.16}$$

The partial derivatives in (7.16) are given by the following expressions:

$$\frac{\partial \breve{g}_{ij}}{\partial w^m} = \sum_{r=1}^{3}\sum_{s=1}^{3} \frac{\partial E^r_i}{\partial w^m}\,g_{rs}\,E^s_j + \sum_{r=1}^{3}\sum_{s=1}^{3} E^r_i\,g_{rs}\frac{\partial E^s_j}{\partial w^m}, \tag{7.17}$$

$$\frac{\partial \breve{g}_{ij}}{\partial J^m_n} = \sum_{r=1}^{3}\sum_{s=1}^{3} \frac{\partial E^r_i}{\partial J^m_n}\,g_{rs}\,E^s_j + \sum_{r=1}^{3}\sum_{s=1}^{3} E^r_i\,g_{rs}\frac{\partial E^s_j}{\partial J^m_n}, \tag{7.18}$$

The metric G_{ij} is another metric entering the nonlinear deformation tensor in (7.4). It is defined in (4.16). By applying (6.2) to (4.16) and taking into account relations (4.14) and (7.12), we obtain

$$\begin{gathered}\hat{G}_{ij}=G_{ij}+\varepsilon\sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\partial\eta_{rs}}{\partial y^m}\,h^m\,J^r_i\,J^s_j\,+\\+\,\varepsilon\sum_{r=1}^{3}\sum_{s=1}^{3}\eta_{rs}\,\nabla_{\!i}h^r\,J^s_j+\varepsilon\sum_{r=1}^{3}\sum_{s=1}^{3}\eta_{rs}\,J^r_i\,\nabla_{\!j}h^s+\ldots.\end{gathered}\tag{7.19}$$

Due to the symmetry of the metric η_{rs}, relation (7.19) can be rewritten as

$$\begin{gathered}\hat{G}_{ij}=G_{ij}+\varepsilon\sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\partial\eta_{rs}}{\partial y^m}\,h^m\,J^r_i\,J^s_j\,+\\+\,\varepsilon\sum_{r=1}^{3}\sum_{s=1}^{3}\eta_{rs}\,(J^r_i\,\nabla_{\!j}h^s+J^r_j\,\nabla_{\!i}h^s)+\ldots.\end{gathered}\tag{7.20}$$

Inversely, the metric η_{rs} can be expressed in terms of the metric G_{pq}:

$$\eta_{rs}=\sum_{p=1}^{3}\sum_{q=1}^{3}G_{pq}\,M^p_r\,M^q_s.\tag{7.21}$$

Differentiating (7.21) with respect to y^m yields

$$\frac{\partial\eta_{rs}}{\partial y^m}=\sum_{n=1}^{3}M^n_m\,\frac{\partial}{\partial x^n}\biggl(\,\sum_{p=1}^{3}\sum_{q=1}^{3}G_{pq}\,M^p_r\,M^q_s\biggr).\tag{7.22}$$

The expression in parentheses in (7.22) is a scalar with respect to coordinate transformations in the physical universe. Therefore, we can replace the partial derivative on the right-hand side of (7.22) by the corresponding covariant derivative:

$$\begin{gathered}\frac{\partial\eta_{rs}}{\partial y^m}=\sum_{n=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3}M^p_r\,M^q_s\,M^n_m\nabla_{\!n}G_{pq}\,+\\+\sum_{n=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3}G_{pq}\,M^p_r\,M^n_m\nabla_{\!n}M^q_s+\sum_{n=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3}G_{pq}\,M^q_s\,M^n_m\nabla_{\!n}M^p_r.\end{gathered}\tag{7.23}$$

Now we substitute (7.23) and (7.21) into (7.20). This yields

$$\begin{gathered}\hat{G}_{ij}=G_{ij}+\varepsilon\sum_{m=1}^{3}\sum_{n=1}^{3}\biggl(\nabla_{\!n}G_{ij}+\sum_{q=1}^{3}\sum_{s=1}^{3}G_{iq}\,J^s_j\,\nabla_{\!n}M^q_s\,+\\+\sum_{p=1}^{3}\sum_{r=1}^{3}G_{pj}\,J^r_i\,\nabla_{\!n}M^p_r\biggr)M^n_m\,h^m\,+\\+\,\varepsilon\sum_{n=1}^{3}\sum_{m=1}^{3}\biggl(\,\sum_{q=1}^{3}G_{iq}\,M^q_m\,\delta^n_j+\sum_{p=1}^{3}G_{pj}\,M^p_m\,\delta^n_i\biggr)\nabla_{\!n}h^m+\ldots.\end{gathered}\tag{7.24}$$

Relation (7.24) can be rewritten as

$$\hat{G}_{ij}=G_{ij}+\varepsilon\sum_{m=1}^{3}\frac{\partial G_{ij}}{\partial y^m}\,h^m+\varepsilon\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{\partial G_{ij}}{\partial J^m_n}\,\nabla_n h^m+\ldots. \tag{7.25}$$

The partial derivatives in (7.25) serve as shorthand notations for the following expressions:

$$\begin{aligned}\frac{\partial G_{ij}}{\partial y^m}=\sum_{n=1}^{3}\biggl(\nabla_n G_{ij}&+\sum_{q=1}^{3}\sum_{s=1}^{3}G_{iq}\,J^s_j\,\nabla_n M^q_s\,+\\&+\sum_{p=1}^{3}\sum_{r=1}^{3}G_{pj}\,J^r_i\,\nabla_n M^p_r\biggr)M^n_m,\end{aligned} \tag{7.26}$$

$$\frac{\partial G_{ij}}{\partial J^m_n}=\biggl(\sum_{q=1}^{3}G_{iq}\,M^q_m\,\delta^n_j+\sum_{p=1}^{3}G_{pj}\,M^p_m\,\delta^n_i\biggr). \tag{7.27}$$

Here and in (7.24), δ^n_i and δ^n_j are the Kronecker deltas (see [44]). Relations (7.26) and (7.27) are derived from (7.24) by using (7.25). These expressions are analogous to (7.14), (7.15), (7.17), and (7.18).

Now we can combine (7.16) and (7.25) to calculate the variation of the relativistic deformation tensor defined in (7.4):

$$\begin{aligned}\hat{u}_{ij}=u_{ij}&+\varepsilon\sum_{m=1}^{3}\frac{\partial u_{ij}}{\partial y^m}\,h^m\,+\\+\,\varepsilon\sum_{m=1}^{3}\frac{\partial u_{ij}}{\partial w^m}\,\dot{h}^m&+\varepsilon\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{\partial u_{ij}}{\partial J^m_n}\,\nabla_n h^m+\ldots.\end{aligned} \tag{7.28}$$

The partial derivatives in (7.28) are given by the following expressions:

$$\begin{aligned}\frac{\partial u_{ij}}{\partial y^m}=-\frac{1}{2}\sum_{n=1}^{3}\biggl(\nabla_n G_{ij}&+\sum_{q=1}^{3}\sum_{s=1}^{3}G_{iq}\,J^s_j\,\nabla_n M^q_s\,+\\&+\sum_{p=1}^{3}\sum_{r=1}^{3}G_{pj}\,J^r_i\,\nabla_n M^p_r\biggr)M^n_m,\end{aligned} \tag{7.29}$$

$$\frac{\partial u_{ij}}{\partial w^m}=\frac{1}{2}\sum_{r=1}^{3}\sum_{s=1}^{3}\biggl(\frac{\partial E^r_i}{\partial w^m}\,g_{rs}\,E^s_j+E^r_i\,g_{rs}\,\frac{\partial E^s_j}{\partial w^m}\biggr), \tag{7.30}$$

$$\begin{aligned}\frac{\partial u_{ij}}{\partial J^m_n}=\frac{1}{2}\sum_{r=1}^{3}\sum_{s=1}^{3}&\biggl(\frac{\partial E^r_i}{\partial J^m_n}\,g_{rs}\,E^s_j+E^r_i\,g_{rs}\frac{\partial E^s_j}{\partial J^m_n}\biggr)-\\-\frac{1}{2}\sum_{q=1}^{3}G_{iq}\,M^q_m\,\delta^n_j&+\frac{1}{2}\sum_{p=1}^{3}G_{pj}\,M^p_m\,\delta^n_i.\end{aligned} \tag{7.31}$$

Relations (7.29), (7.30), and (7.31) are derived by using (7.17), (7.18), (7.26), and (7.27). The partial derivatives still remaining on the right-hand sides of (7.30) and

(7.31) are given by (7.14) and (7.15).

Note that the nonlinear relativistic deformation tensor (7.4), denoted by $\mathbf{u}$, enters as an argument into the potential energy density $\mathcal{F}(y, J, \mathbf{v}, \mathbf{u})$ in the Lagrangian density (4.10).

8. Distortion tensor.

The Jacobian matrices M^r_m and J^i_j were already used above. They are given by (4.14) and are mutually inverse. The former is known as the direct distortion tensor, while the latter is known as the inverse distortion tensor. In the linear theory of elasticity (see [47]), the deformation tensor is defined as the symmetric part of the distortion tensor. However, the present case is different since we deal with the nonlinear theory.

We now apply the small variations (6.2) to the inverse distortion tensor J^i_j. Differentiating (6.2) with respect to x^j and taking into account (4.14) and (7.12), we obtain

$$\hat{J}^i_j = J^i_j + \varepsilon\,\nabla_{\!j} h^i; \tag{8.1}$$

compare this result with formula (7.12) in [17]. An analogous relation is available for the direct distortion tensor M^r_m (see (6.5)). Relation (8.1) is simpler than (6.5). Consequently, the inverse distortion tensor J is preferable as an argument of the potential energy density $\mathcal{F}(y, J, \mathbf{v}, \mathbf{u})$ in (4.10).

9. Small variations of the kinetic energy.

The kinetic part $\mathcal{K}$ of the Lagrangian density of matter is given by the first relation in (6.8). The term ρ therein represents the rest mass density defined in (4.17). By applying (6.2) to ρ, we previously obtained relation (6.9), which can be rewritten as

$$\begin{aligned}\hat{\rho} = \rho\,\biggl(1 + \varepsilon\sum^3_{m=1}\sum^3_{n=1}&\frac{\nabla_{\!n}(\rho\sqrt{\det g}/|\det J|)}{\rho\sqrt{\det g}/|\det J|}\,M^n_m\,h^m\,+\\ &+\varepsilon\sum^3_{m=1}\sum^3_{n=1}M^n_m\,\nabla_{\!n}h^m\biggr)+\ldots.\end{aligned} \tag{9.1}$$

Since $|\det J| = \pm\det J$, relation (9.1) can be further simplified to

$$\begin{aligned}\hat{\rho} = \rho\,\biggl(1 + \varepsilon\sum^3_{m=1}\sum^3_{n=1}&\frac{\nabla_{\!n}(\rho\sqrt{\det g}/\det J)}{\rho\sqrt{\det g}/\det J}\,M^n_m\,h^m\,+\\ &+\varepsilon\sum^3_{m=1}\sum^3_{n=1}M^n_m\,\nabla_{\!n}h^m\biggr)+\ldots.\end{aligned} \tag{9.2}$$

Relation (9.2) is structurally analogous to (7.16), (7.24), and (7.28). Consequently, it can be rewritten as

$$\hat{\rho} = \rho + \varepsilon\sum^3_{m=1}\frac{\partial\rho}{\partial y^m}\,h^m + \varepsilon\sum^3_{n=1}\sum^3_{m=1}\frac{\partial\rho}{\partial J^m_n}\,\nabla_{\!n}h^m + \ldots. \tag{9.3}$$

The partial derivatives in (9.3) serve as shorthand notations for the following expressions:

$$\frac{\partial\rho}{\partial y^m}=\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\nabla_n(\rho\sqrt{\det g}/\det J)}{\rho\sqrt{\det g}/\det J}\,M^n_m,\tag{9.4}$$

$$\frac{\partial\rho}{\partial J^m_n}=M^n_m.\tag{9.5}$$

Relations (9.4) and (9.5) are derived from (9.2). They extend the list of analogous expressions (7.14), (7.15), (7.17), (7.18), (7.26), (7.27), (7.29), (7.30), (7.31), (8.1).

Apart from ρ, the expression for $\mathcal{K}$ in (6.8) contains a square-root factor. By applying (6.2) and (6.3) to this factor, we previously obtained relation (6.10). Since this relation is structurally analogous to (7.13), it can be rewritten as

$$\begin{aligned}c_{\text{br}}^2\sqrt{g_{00}-\frac{|\hat{\mathbf{v}}|^2}{c_{\text{br}}^2}}=c_{\text{br}}^2\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\,+\\+\,\varepsilon\sum_{m=1}^{3}A_m\,\dot{h}^m+\varepsilon\sum_{m=1}^{3}\sum_{n=1}^{3}B^n_m\,\nabla_n h^m+\ldots\,.\end{aligned}\tag{9.6}$$

The symbols A_m and B^n_m in (9.6) denote the following expressions:

$$A_m=\sum_{s=1}^{3}\frac{v_s\,M^s_m}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}},\qquad\qquad B^n_m=\sum_{s=1}^{3}\frac{v_s\,M^s_m\,v^n}{\sqrt{g_{00}-\dfrac{|\mathbf{v}|^2}{c_{\text{br}}^2}}}.\tag{9.7}$$

Relations (9.7) are derived from (6.10) and extend the list of analogous expressions.

Now we can apply (6.2) to the kinetic part $\mathcal{K}$ of the Lagrangian density of matter defined by the first relation in (6.8):

$$\hat{\mathcal{K}}=\mathcal{K}+\varepsilon\sum_{m=1}^{3}\frac{\partial\mathcal{K}}{\partial y^m}\,h^m+\varepsilon\sum_{m=1}^{3}\frac{\partial\mathcal{K}}{\partial w^m}\,\dot{h}^m+\varepsilon\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{\partial\mathcal{K}}{\partial J^m_n}\,\nabla_n h^m+\ldots\,.\tag{9.8}$$

The partial derivatives in (9.8) are given by the following expressions:

$$\frac{\partial\mathcal{K}}{\partial y^m}=c_{\text{br}}^2\,\frac{\partial\rho}{\partial y^m}\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}},\tag{9.9}$$

$$\frac{\partial\mathcal{K}}{\partial w^m}=\rho\,A_m,\tag{9.10}$$

$$\frac{\partial\mathcal{K}}{\partial J^m_n}=\rho\,B^n_m+c_{\text{br}}^2\,\frac{\partial\rho}{\partial J^m_n}\sqrt{g_{00}-\frac{|\mathbf{v}|^2}{c_{\text{br}}^2}}\tag{9.11}$$

The terms A_m and B^n_m, along with the partial derivatives appearing on the right-hand sides of (9.9), (9.10), and (9.11), can be made explicit sequentially. Specifically, A_m and B^n_m are given by (9.7), while relations (9.4) and (9.5) are applied thereafter.

10. Small variations of the potential energy.

The potential part $-\mathcal{F}$ of the Lagrangian density of matter is the second term on the right-hand side of relation (4.10). Although it is not given explicitly, we can apply (6.2) to it to obtain an expansion analogous to (9.8):

$$\hat{\mathcal{F}}=\mathcal{F}+\varepsilon\sum_{m=1}^{3}\frac{\partial\mathcal{F}}{\partial y^m}\,h^m+\varepsilon\sum_{m=1}^{3}\frac{\partial\mathcal{F}}{\partial w^m}\,\dot{h}^m+\varepsilon\sum_{n=1}^{3}\sum_{m=1}^{3}\frac{\partial\mathcal{F}}{\partial J^m_n}\,\nabla_n h^m+\ldots.\tag{10.1}$$

The partial derivatives on the left-hand side of (10.1) do not correspond directly to the arguments of the initial function $\mathcal{F}$ in (4.10). They are given by the expressions

$$\frac{\partial\mathcal{F}}{\partial y^m}=\frac{\partial\mathcal{F}}{\partial y^m}+\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial\mathcal{F}}{\partial u_{ij}}\,\frac{\partial u_{ij}}{\partial y^m},\tag{10.2}$$

$$\frac{\partial\mathcal{F}}{\partial w^m}=-\sum_{n=1}^{3}\frac{\partial\mathcal{F}}{\partial v^n}\,M^n_m+\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial\mathcal{F}}{\partial u_{ij}}\,\frac{\partial u_{ij}}{\partial w^m},\tag{10.3}$$

$$\frac{\partial\mathcal{F}}{\partial J^m_n}=\frac{\partial\mathcal{F}}{\partial J^m_n}-\sum_{s=1}^{3}\frac{\partial\mathcal{F}}{\partial v^s}M^s_m\,v^n+\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial\mathcal{F}}{\partial u_{ij}}\,\frac{\partial u_{ij}}{\partial J^m_n}.\tag{10.4}$$

In contrast, the partial derivatives of $\mathcal{F}$ appearing on the right-hand sides of these relations are standard; they correspond to the arguments of this function in (4.10).

In deriving (10.2), (10.3), and (10.4), we used relations (6.7), (7.28), and (8.1). The partial derivatives inherited from them can be written explicitly by consecutively applying (7.29), (7.30), (7.31), and then (7.14) and (7.15).

11. Stationary action principle at boundaries.

The principle of stationary action (see [42]) in our case states that the action integral S in (4.1) is stationary under any smooth variations $h^i(t,x^1,x^2,x^3)$ with compact support defined in (6.2). The gravitational part of the Lagrangian density $\mathcal{L}_{\text{gr}}$ in (4.2) does not depend on h^i. Consequently, we can omit it from our considerations and deal exclusively with the action integral S_{mat} in (4.3) and (4.5).

According to (9.8) and (10.1), the variation of the Lagrangian density $\mathcal{L}_{\text{mat}}$ in (4.10) takes the form of a sum with three leading terms:

$$\begin{gathered}\delta\mathcal{L}_{\text{mat}}=\hat{\mathcal{L}}_{\text{mat}}-\mathcal{L}_{\text{mat}}=\\=\varepsilon\sum_{m=1}^{3}\mathcal{A}_m\,h^m+\varepsilon\sum_{m=1}^{3}\mathcal{B}_m\,\dot{h}^m+\varepsilon\sum_{n=1}^{3}\sum_{m=1}^{3}\mathcal{C}^n_m\,\nabla_n h^m+\ldots.\end{gathered}\tag{11.1}$$

Due to (4.10), the coefficients $\mathcal{A}_m$ and $\mathcal{B}_m$ in (11.1) are given by the relations

$$\mathcal{A}_m=\frac{\partial\mathcal{K}}{\partial y^m}-\frac{\partial\mathcal{F}}{\partial y^m},\qquad\qquad \mathcal{B}_m=\frac{\partial\mathcal{K}}{\partial w^m}-\frac{\partial\mathcal{F}}{\partial w^m}.\tag{11.2}$$

The coefficients $\mathcal{C}^n_m$ in (11.1) are given by an analogous relation:

$$\mathcal{C}^n_m=\frac{\partial\mathcal{K}}{\partial J^m_n}-\frac{\partial\mathcal{F}}{\partial J^m_n},\tag{11.3}$$

Relations (11.2) and (11.3) are derived by using (9.8) and (10.1).

Now we apply (11.1) to the matter integral (4.4). This yields

$$\begin{gathered}\delta L_{\text{mat}}=\int\delta\mathcal{L}_{\text{mat}}\sqrt{\det g}\,d^3x=\varepsilon\int\sum_{m=1}^{3}\mathcal{A}_m\,h^m\sqrt{\det g}\,d^3x\,+\\+\,\varepsilon\int\sum_{m=1}^{3}\mathcal{B}_m\,\dot h^m\sqrt{\det g}\,d^3x+\varepsilon\int\sum_{n=1}^{3}\sum_{m=1}^{3}\mathcal{C}^n_m\,\nabla_n h^m\sqrt{\det g}\,d^3x+\ldots.\end{gathered}\tag{11.4}$$

For brevity, we write (11.4) as

$$\delta L_{\text{mat}}=\varepsilon I_1+\varepsilon I_2+\varepsilon I_3+\ldots.\tag{11.5}$$

Then, according to (4.5), we integrate (11.5) with respect to time:

$$\delta S_{\text{mat}}=\varepsilon\int I_1\,dt+\varepsilon\int I_2\,dt+\varepsilon\int I_3\,dt+\ldots.\tag{11.6}$$

Recall that our elastic medium is subdivided into two domains M^+ and M^- (see Fig. 5.1). Consequently, the integrals I_1, I_2, and I_3 take the form

$$I_1=\int\limits_{M^+}\sum_{m=1}^{3}\mathcal{A}_m\,h^m\sqrt{\det g}\,d^3x+\int\limits_{M^-}\sum_{m=1}^{3}\mathcal{A}_m\,h^m\sqrt{\det g}\,d^3x,\tag{11.7}$$

$$I_2=\int\limits_{M^+}\sum_{m=1}^{3}\mathcal{B}_m\,\dot h^m\sqrt{\det g}\,d^3x+\int\limits_{M^-}\sum_{m=1}^{3}\mathcal{B}_m\,\dot h^m\sqrt{\det g}\,d^3x,\tag{11.8}$$

$$\begin{gathered}I_3=\int\limits_{M^+}\sum_{n=1}^{3}\sum_{m=1}^{3}\mathcal{C}^n_m\,\nabla_n h^m\sqrt{\det g}\,d^3x\,+\\+\int\limits_{M^-}\sum_{n=1}^{3}\sum_{m=1}^{3}\mathcal{C}^n_m\,\nabla_n h^m\sqrt{\det g}\,d^3x.\end{gathered}\tag{11.9}$$

The integrals in (11.8) can be transformed by using the Reynolds transport theorem for domains with moving boundaries (see [48]), which yields

$$\begin{gathered}\int\limits_{M^+}\sum_{m=1}^{3}\mathcal{B}_m\,\dot h^m\sqrt{\det g}\,d^3x=\frac{d}{dt}\int\limits_{M^+}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\sqrt{\det g}\,d^3x\,-\\-\int\limits_{M^+}\sum_{m=1}^{3}\frac{\partial(\mathcal{B}_m\sqrt{\det g}\,)}{\partial t}\,h^m\,d^3x-\int\limits_{\partial M^+}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,(\mathbf{u},\mathbf{n})\,dS.\end{gathered}\tag{11.10}$$

$$\begin{gathered}\int\limits_{M^-}\sum_{m=1}^{3}\mathcal{B}_m\,\dot h^m\sqrt{\det g}\,d^3x=\frac{d}{dt}\int\limits_{M^-}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\sqrt{\det g}\,d^3x\,-\\-\int\limits_{M^-}\sum_{m=1}^{3}\frac{\partial(\mathcal{B}_m\sqrt{\det g}\,)}{\partial t}\,h^m\,d^3x-\int\limits_{\partial M^-}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,(\mathbf{u},\mathbf{n})\,dS.\end{gathered}\tag{11.11}$$

The first term on the right-hand side of (11.10) vanishes when integrating with respect to time in (11.6). The second term contributes to the derivation of the Lagrange equations within the domain M^{+}, which were already studied in [17]. Therefore, we retain only the boundary integral in (11.10) for further consideration. The same arguments apply to (11.11):

$$\int\limits_{M^{+}}\sum_{m=1}^{3}\mathcal{B}_m\,\dot{h}^m\,\sqrt{\det g}\;d^3x=-\!\!\int\limits_{\partial M^{+}}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,(\mathbf{u},\mathbf{n})\,dS+\ldots\,,\tag{11.12}$$

$$\int\limits_{M^{-}}\sum_{m=1}^{3}\mathcal{B}_m\,\dot{h}^m\,\sqrt{\det g}\;d^3x=-\!\!\int\limits_{\partial M^{-}}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,(\mathbf{u},\mathbf{n})\,dS+\ldots\,.\tag{11.13}$$

Here, $\mathbf{u}$ is the velocity vector of the surface element and $\mathbf{n}$ is the unit vector of its outward normal. In terms of their components, we can write (11.12) and (11.13) as

$$\int\limits_{M^{+}}\sum_{m=1}^{3}\mathcal{B}_m\,\dot{h}^m\,\sqrt{\det g}\;d^3x=-\!\!\int\limits_{\partial M^{+}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,u^s\,n_s\,dS+\ldots\,,\tag{11.14}$$

$$\int\limits_{M^{-}}\sum_{m=1}^{3}\mathcal{B}_m\,\dot{h}^m\,\sqrt{\det g}\;d^3x=-\!\!\int\limits_{\partial M^{-}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,u^s\,n_s\,dS+\ldots\,.\tag{11.15}$$

Next, we consider the integrals in (11.9) and apply the Ostrogradsky-Gauss theorem (see [49]) to them. This yields

$$\int\limits_{M^{+}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^s_m\,\nabla_{\!s}h^m\sqrt{\det g}\;d^3x=\int\limits_{\partial M^{+}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^s_m\,h^m\,n_s\,dS-\ldots\,,\tag{11.16}$$

$$\int\limits_{M^{-}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^s_m\,\nabla_{\!s}h^m\sqrt{\det g}\;d^3x=\int\limits_{\partial M^{-}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^s_m\,h^m\,n_s\,dS-\ldots\,.\tag{11.17}$$

The non-boundary terms on the right-hand sides of (11.16) and (11.17) are intentionally omitted, as they contribute to the derivation of the Lagrange equations within the bulk of the domains. As for the integrals in (11.7), they do not require any transformations and thus yield no boundary terms, contributing only to the Lagrange equations.

Collecting the boundary integrals from (11.14), (11.15), (11.16), and (11.17), we find their total effect on the variation of the action in (11.6). According to the principle of stationary action (see [42]), this total effect must vanish. Hence,

$$\begin{gathered}-\!\!\int\limits_{\partial M^{+}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,u^s\,n_s\,dS+\!\!\int\limits_{\partial M^{+}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^s_m\,h^m\,n_s\,dS-\\-\!\!\int\limits_{\partial M^{-}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}_m\,h^m\,u^s\,n_s\,dS+\!\!\int\limits_{\partial M^{-}}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^s_m\,h^m\,n_s\,dS=0.\end{gathered}\tag{11.18}$$

Note that the boundaries ∂M^{+} and ∂M^{-} represent the same surface σ that serves as the interface between the elastic media (see Fig. 5.1). However, by writing ∂M^{+} and ∂M^{-} in (11.18), we treat them as two coinciding surfaces with opposite orientations and opposite unit normal vectors. By passing to the common normal vector $\mathbf{n}$ shown in Fig. 5.1, which is outward for M^{-} and inward for M^{+}, we can rewrite (11.18) as

$$\begin{aligned}&\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}^{+}_{m}\,h^{m}\,u^{s}\,n_{s}\,dS-\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^{s+}_{m}\,h^{m}\,n_{s}\,dS\;=\\&=\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}^{-}_{m}\,h^{m}\,u^{s}\,n_{s}\,dS-\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^{s-}_{m}\,h^{m}\,n_{s}\,dS.\end{aligned}\tag{11.19}$$

The plus and minus superscripts of the coefficients $\mathcal{B}^{+}_{m}$, $\mathcal{C}^{s+}_{m}$, $\mathcal{B}^{-}_{m}$, and $\mathcal{C}^{s-}_{m}$ indicate which of the two media they belong to. These coefficients are given by the same relations (11.2) and (11.3) as before, but correspond to different Lagrangian densities defined in (5.3) and (5.4).

12. Boundary conditions.

We assume that the media remain in continuous contact, i. e., without forming cracks or voids at the interface σ. This assumption leads to the following boundary condition for the components of the velocity vector $\mathbf{v}$:

$$\sum_{i=1}^{3}v^{i}_{+}\,n_{i}\,\biggr|_{\sigma}=\sum_{i=1}^{3}v^{i}_{-}\,n_{i}\,\biggr|_{\sigma}.\tag{12.1}$$

Relation (12.1) implies that the normal components of the velocity vector $\mathbf{v}$ on both sides of the interface σ are equal. The velocity vector $\mathbf{u}$ of the interface itself must have the same normal component to maintain its integrity. Consequently, (12.1) can be extended to

$$\sum_{i=1}^{3}v^{i}_{+}\,n_{i}\,\biggr|_{\sigma}=\sum_{i=1}^{3}v^{i}_{-}\,n_{i}\,\biggr|_{\sigma}=\sum_{i=1}^{3}u^{i}\,n_{i}\,\biggr|_{\sigma}.\tag{12.2}$$

If the tangential components of the vector $\mathbf{v}$ on opposite sides of the interface σ are not necessarily equal, the boundary condition (12.1) is called the ideal slip condition. Conversely, if the vector field $\mathbf{v}$ is continuous at the interface, we write

$$v^{i}_{+}\,\biggr|_{\sigma}=v^{i}_{-}\,\biggr|_{\sigma}.\tag{12.3}$$

The boundary condition (12.3) is called the perfect contact condition.

In each of these cases, relation (12.2) follows directly from (12.1) or from (12.3). Consequently, relation (11.19) can be rewritten as

$$\begin{aligned}&\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}^{+}_{m}\,h^{m}\,v^{s}_{+}\,n_{s}\,dS-\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^{s+}_{m}\,h^{m}\,n_{s}\,dS\;=\\&=\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{B}^{-}_{m}\,h^{m}\,v^{s}_{-}\,n_{s}\,dS-\int\limits_{\sigma}\sum_{s=1}^{3}\sum_{m=1}^{3}\mathcal{C}^{s-}_{m}\,h^{m}\,n_{s}\,dS.\end{aligned}\tag{12.4}$$

Relation (12.4) holds for arbitrary variations $h^m(t,x^1,x^2,x^3)$ in (6.2). Consequently, it leads to the following boundary conditions:

$$\sum_{s=1}^{3}\Bigl(\mathcal{B}^{+}_{m}\,v^{s}_{+}-\mathcal{C}^{s+}_{m}\Bigr)n_s\Bigr|_{\sigma}=\sum_{s=1}^{3}\Bigl(\mathcal{B}^{-}_{m}\,v^{s}_{-}-\mathcal{C}^{s-}_{m}\Bigr)\,n_s\Bigr|_{\sigma}. \tag{12.5}$$

Based on the number of constraints, boundary conditions (12.5) correspond to the case of perfect contact (12.3), though relations (12.3) were not used explicitly. They could have been implicitly assumed by choosing the same variations $h^m(t,x^1,x^2,x^3)$ from (6.2) on both sides of the interface σ. Consequently, the case of ideal slip (12.1) should be investigated separately in a subsequent paper.

13. Conclusions.

Boundary conditions (12.5) are the main result of the present paper. Along with the conditions in (12.3), they can be applied to the Lagrange equations for relativistic nonlinear elastic media derived in [17] within the framework of the 3D-brane universe model (see [2] or [3]).

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15. Dedication.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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